Tutorial 1: Asymptotic Notation

Example Solutions

1. Prove the following facts stated in Lecture Note 2.
(a) For any constant a > 0 in \( \mathbb{R} \): \( f_1 = O(g_1) \implies a f_1 = O(g_1) \).
We are given that there are constants \( n_0 \in \mathbb{N} \) and \( c > 0 \) such that
\[
0 \leq f_1(n) \leq c g_1(n)
\]
for all \( n \geq n_0 \). We want to find constants \( n_1 \) and \( c_1 > 0 \) such that
\[
0 \leq a f_1(n) \leq c_1 g_1(n)
\]
for all \( n \geq n_1 \). Since \( a > 0 \) we have
\[
0 \leq a f_1(n) \leq a c g_1(n)
\]
so we can take \( n_1 = n_0 \) and \( c_1 = a c \).
(b) \( f_1 = O(g_1) \) and \( f_2 = O(g_2) \implies f_1 + f_2 = O(g_1 + g_2) \).
Here we are given that there are \( n_1, n_2 \in \mathbb{N} \) and \( c_1, c_2 > 0 \) such that
\[
0 \leq f_1(n) \leq c_1 g_1(n), \text{ for all } n \geq n_1;
\]
\[
0 \leq f_2(n) \leq c_2 g_2(n), \text{ for all } n \geq n_2.
\]
So if we put \( n_0 = \max(n_1, n_2) \) then the two pairs of inequalities hold at the same
time and we can add them to obtain
\[
0 \leq f_1(n) + f_2(n) \leq c_1 g_1(n) + c_2 g_2(n),
\]
for all \( n \geq n_0 \). All that remains now is to take \( c = \max(c_1, c_2) \) and observe that
\[
c_1 g_1(n) + c_2 g_2(n) \leq c_1 g(n) + c_2 g(n) = c(g(n) + g(n)).
\]
In the above we have used the standard fact that if \( a_1 \leq b_1 \) and \( a_2 \leq b_2 \) then
\( a_1 + a_2 \leq b_1 + b_2 \)

2. Here we consider a function of the form
\[
g(n) = f(n) + \frac{a_1}{n} + \frac{a_2}{n^2}
\]
where \( f \) is some other function with \( f(n) \geq 0 \) for all \( n \) and \( a_1, a_2 \) are non-negative constants.
(a) As \( n \) increases \( 1/n \) and \( 1/n^2 \) both decrease. So they have their biggest value
when \( n = 1 \). So \( g(n) \leq f(n) + a_1 + a_2 \) (what goes wrong if either of the constants
is negative\( ? \)). Well any additive constant can be replaced by 1 for the purposes
of asymptotic notation. Thus \( g = O(f + 1) \). (More formally, \( f(n) + a_1 + a_2 = O(f(n)) + O(1) + O(1) = O(f(n) + 1) \).)

3. (a) True. Just take \( c = 1, n_0 = 0 \) in the definition of big-\( \Omega \).
(b) True. Just take \( c = 1, n_0 = 0 \) in the definition of big-O.
(c) True. Since \( T_A(n) \) is the maximum runtime of \( A \) over all inputs of size \( n \),
it follows that all inputs of size \( n \) have runtime at most \( T_A(n) \). The claim now
follows since \( T_A(n) \leq U(n) \) for all \( n \).
(d) False. It is perfectly possible that for some inputs of size \( n \) the runtime of \( A \)
is better than the worst possible, i.e., better than \( T_A(n) \). Thus there could well
be inputs of size \( n \) for which the runtime is less than \( U(n) \). (As pointed out in the
notes linear search is an example of such an algorithm.)
(e) True. By the definition of \( T_A \) there is at least one input \( I \) of size \( n \) that has
runtime \( T_A(n) \). Since \( U(n) \leq T_A(n) \) for all \( n \) the claim holds for the input \( I \) and
possibly others of course but we cannot decide this without further information
about \( A \).

4. (a) Each of the last \( [n/2] \) factors in \( 1 \times 2 \times \ldots \times n \) is at least as big as \( [n/2] \) and
so the lower bound on \( n! \) follows. On the other hand, each of the \( n \) factors in
\( 1 \times 2 \times \ldots \times n \) is no larger than \( n \) and so the upper bound follows.
(b) Taking logarithms in the bounds of the preceding part we have
\( [n/2] \log [n/2] \leq \log n! \leq n \log n \). The upper bound shows immediately that \( \log n! = O(n \log n) \); we take
\( n_0 = 1 \) and \( c = 1 \) in the definition of \( O \).
From the lower bound we have \( \log n! \geq [n/2] \log [n/2] \geq (n/2) \log (n/2) \). We can proceed
in various ways here, one is to note that \( n/2 \geq \sqrt{n} \) for \( n \geq 4 \) and so for all such
values we have \( \log n! \geq \log (n/2) \geq (n/2) \log \sqrt{n} = (n/4) \log n \). Taking \( n_0 = 4 \) and
\( c = 1/4 \) in the definition of \( \Omega \) we see that \( \log n! = \Omega(n \log n) \).
Since \( \log n! = O(n \log n) \) and \( \log n! = \Omega(n \log n) \) we have \( \log n! = \Theta(n \log n) \) as claimed.

5. (a) We use the standard assumption that each line of the algorithm takes constant
\( O(1) \) time. The first loop is executed \( O(n) \) times. The body of the loop executes
lines 2 and 5 which take constant time as well as the loop at line 3. This loop
itself is executed \( O(n) \) times and its body takes constant time. Thus the body of
the first loop takes time \( O(1) + O(n) = O(n) \). It follows that the overall time is
\( O(n)O(n) = O(n^2) \).
(b) Let \( B = 4 \) (in fact any array that has 1 as an entry will do). The two loops
are each executed \( n \) times since \( \text{found} \) is always set to \( \text{True} \) by the inner loop so
that in line 5 we do not return early. So the runtime is at least $cn^2$ for a constant $c$, i.e., $O(n^2)$.

Of course a sensible modification of the algorithm would be to exit the inner loop on lines 3 and 4 as soon as found is set to True. If the algorithm is modified in this way then $B = \{2, 2, \ldots, 2, 1\}$ produces the worst case behaviour. Note that this observation is made here to further understanding, it is not required as part of the answer to the question (so bear this in mind for the exam). Can you think of other possible arrays that produce the worst case behaviour? Does it have to be at the end?

If you have time you might like to implement the simple algorithm modified to have a counter that is initialized to 0 and is increased by 1 every time one either of the loops is executed. Get the algorithm to return this counter as the result. Don’t spend a great deal of time on this, it can be done in an appropriate language within a matter of minutes.

(c) Yes the runtime is $\Theta(n^2)$ since the runtime of the algorithm is both $O(n^2)$ and $\Omega(n^2)$.

(d) Sort array $B$ using any algorithm with $O(n \lg n)$ worst case runtime (e.g., mergesort, we will study this in the course if you have not yet come across it). Now replace the inner loop with binary search, so that this runs in time $O(\lg n)$.

Thus the two loops together now run in time $O(n)O(\lg n) = O(n \lg n)$. Hence the overall runtime is $O(n \lg n) + O(n \lg n) = O(n \lg n)$.

6. (a) For $S = \{\{x_1, \neg x_2\}, \{x_2\}\}$ we have

\[
R(S) = S \cup \{\{x_1\}\} \\
R^2(S) = R(S).
\]

Since the empty clause is not found we conclude that the formula is satisfiable (e.g., by setting $x_1 = \text{True}$ and $x_2 = \text{False}$).

For $S = \{\{x_1, \neg x_2\}, \{\neg x_2\}, \{x_2\}\}$ we have

\[
R(S) = S \cup \{\{\neg x_2\}\}, \{x_1\}\} \\
R^2(S) = R(S) \cup \{\}\}.
\]

Since we have found the empty clause we do not need to carry on (this is an obvious optimisation of the algorithm). We conclude that the formula is not satisfiable. Check this by trying all 4 possible assignments. Quite often resolution will produce the empty clause early on, obviously this is much better than trying all $2^n$ possible assignments.

(b) There are finitely many clauses so the set of resolvents cannot keep growing, eventually it must repeat.

(c) The first important point to observe is that the resolvent of two clauses each with at most two literals is another such clause. So we should find out how many such clauses there are (any good upper bound would do but the situation is so simple that we can be precise). We have $2n$ distinct literals ($n$ variables and $n$ negated ones). Thus the number of clauses with one literal is $2n$. The number of clauses with two literals is $\binom{2n}{2} = n(n-1)/2$. Of course there is only one clause with no literals. So the number of clauses under consideration is $1 + 2n + n(n-1)/2 = O(n^2)$.

At each stage we thus have $O(n^2)$ clauses and as we compare pairs there are $O(n^2) \times O(n^2) = O(n^4)$ to consider. Since comparison is assumed to be constant time this gives a bound for the time of each stage. The number of stages cannot be more than the total number of clauses (consisting of at most two literals) and is thus $O(n^2)$. So the overall runtime is bounded by $O(n^2) \times O(n^4) = O(n^6)$.

(d) The problem with having clauses that consist of three literals is that their resolvents can lead to clauses with more than three literals. These then can lead to bigger ones etc. In the worst case we obtain exponentially many clauses.

It is worth noting that nobody knows of an algorithm that is not exponential in the worst case. It is not expected that there is one but to date no proof is known. An answer to this question would be of very great significance (and win a large cash prize). The satisfiability problem is the archetypal example of an NP-complete problem. The study of these has kept many computer scientists busy and gainfully employed since the 1970's.