

# Small world networks

Social and Technological Networks

Rik Sarkar

University of Edinburgh, 2019.

# Milgram's experiment

- Take people from random locations in USA
- Ask them to deliver a letter to a fixed person in Massachusetts
- You can only forward the letter to someone you know
- Question: How many hops do the letters take to get to destination?

# Results

- Out of 296 letters, only 64 completed
- Number of hops varied between 2 and 10
- Mean number of hops 6
- There were a few people that were the last hop in most cases

# Discussion of experiment

# Discussion of experiment

- Short paths exist between pairs (small diameter)
- More surprisingly, people find these short paths
- Without knowing the entire network
- *Decentralized search*
- Analogous to routing without a routing table
- People use a “greedy” strategy
- Forward to the friend nearest to the destination

# Recent results

- Milgrams results reproduced on better data
- Use online data (Livejournal, facebook)
- Containing approximate locations
- Simulate the process of forwarding letters
- Results similar to original experiment
- Relatively short diameter, successful decentralized search

# In popular culture

- Erdos distance
- Kevin bacon distance

# Definition of small worlds

- Small diameter
- Large clustering coefficient
  - Related to homophily — similar people connect to each-other
  - “Similar”: close in some coordinate value (or metric)
- Supports decentralized search
  - People find short paths without knowing the entire network



# Model 1: Watts and Strogatz

Nature 1998

- Parameters  $n, k, p$       $n > k > \ln n$ 
  - Often  $k$  is taken to be a constant in practice with the idea that people cannot have infinitely large friend-circles
- Put nodes in a ring of size  $n$
- Connect each to  $k/2$  neighbors on each side

# Model 1: Watts and Strogatz

- What are the diameter and CC?

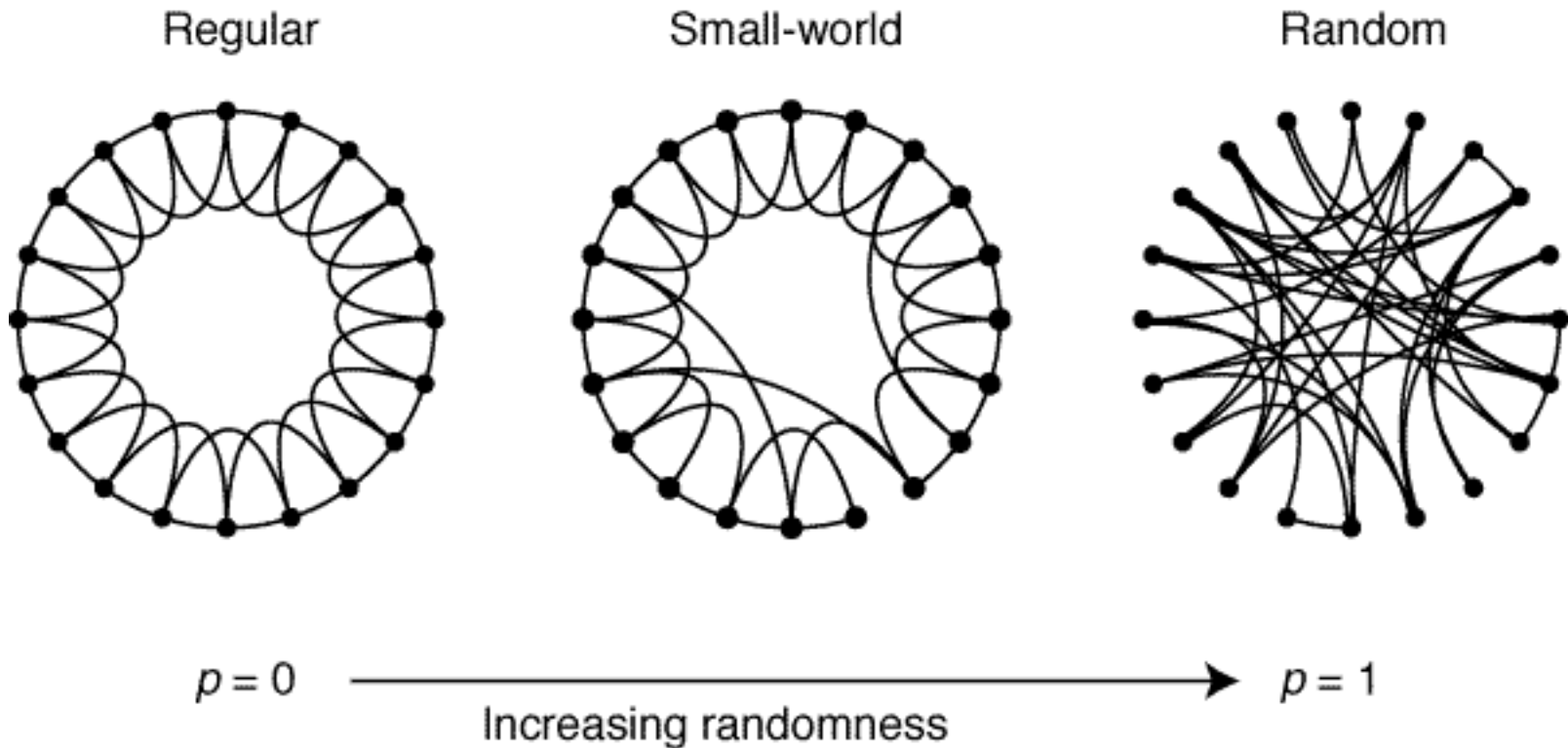
# Model 1: Watts and Strogatz

Nature 1998

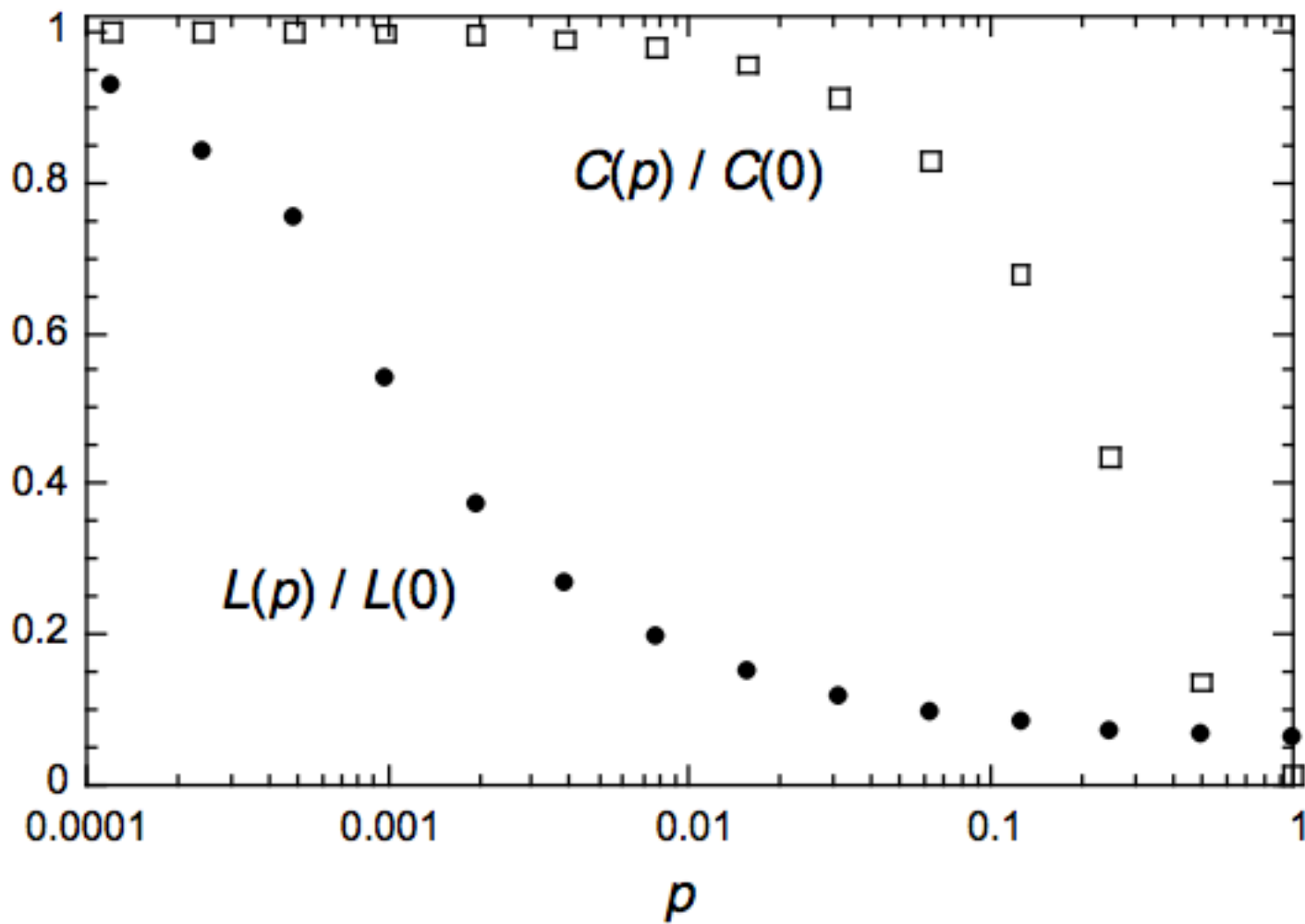
- Parameters  $n, k, p$   $n > k > \ln n$ 
  - Often  $k$  is taken to be a constant in practice with the idea that people cannot have infinitely large friend-circles
- Put nodes in a ring of size  $n$
- Connect each to  $k/2$  neighbors on each side
- With probability  $p$  rewire each edge of a vertex to a random vertex

# Small world

- In between random and structured



# Small world



# Properties

- Average clustering coefficient per vertex bounded away from zero
  - In other words: at least a constant
- Connected: sufficient random edges + regular edges
- Short diameter

Watts-strogatz model does not explain  
milgram's experiment

# Watts-strogatz model does not explain milgram's experiment

- Milgram's experiment was on 2D plane



# A 2D version of Watts-Strogatz

- Take a 2D grid
- *Add* some random edges
  - E.g. from each vertex, add an edge to a random other vertex
- Theorem: Then there is no local/decentralised algorithm that delivers the message in less than  $\text{poly}(n)$  messages.

# Watts-strogatz model does not explain milgram's experiment

- Milgram's experiment was on 2D plane
- Watts-strogatz does not support decentralized search -- takes  $(\text{poly}(\log n))$  steps to destination

# Proof (Optional): Decentralized search in random link networks

- Consider  $s$  and  $t$  separated by  $\Omega(n)$  hops
- Take ball  $B$  of extrinsic radius around  $n^{2/3}$   $t$ 
  - There are  $O(n^{2/3})^2$  nodes in  $B$
- When we are already at distance  $n^{2/3}$  (on the edge of  $B$ )
  - A long link can help only if it falls inside  $B$
- Otherwise we take a step along a short link
- What is the probability that a random link from  $s$  hits  $B$ ?
- This is  $\sim O((n^{2/3})^2/n^2) = O(n^{-2/3})$
- The expected number of steps before getting a useful long link is :  $\Omega(n^{2/3})$

# Model 2 : Kleinberg's model

STOC 2000, Nature 2000, ICM 2006

- Idea: Long links are not helping much
  - Getting closer to the destination does not increase the chances of getting a long link close to destination.
- Make the probability of a long link sensitive to the distance
  - Nearby nodes are more likely to have a long link

# Model 2 : Kleinberg's model

- Suppose  $d(u, v)$  is the extrinsic distance between nodes  $u$  and  $v$  in the plane
- Then  $u$  connects its long link to  $v$
- with probability  $\propto \frac{1}{d(u, v)^\alpha}$

# Kleinberg's model

- Links to nearby nodes are more likely
  - A node knows more people locally
  - With increasing distance, it knows fewer and fewer people
  - At the largest scale it knows only a handful
  - More representative of how people have their contacts spread
- We want to show that the model permits short paths to be found

- Kleinberg's model supports  $(\log(n))^2$  step route
  - Show that the route can be divided into  $O(\log n)$  phases
  - Each phase takes  $O(\log n)$

# The proportionality constant

$$\Pr[(u, v)] = \frac{1}{\gamma} \frac{1}{d(u, v)^\alpha}$$

$$\alpha = 2 \Rightarrow \gamma = \Theta(\ln n)$$

- Sketch of proof (optional): Take rings of thickness 1 at distances 1,2,3...
- The number of nodes at distance  $d \sim \Theta(d)$
- Thus from any node:

$$\frac{1}{\gamma} \sum_{d=1}^n d^{-2} \Theta(d) = 1$$

$$\frac{1}{\gamma} \Theta \left( \sum_{d=1}^n \frac{1}{d} \right) = 1$$

$$\Rightarrow \frac{1}{\gamma} \Theta(\ln n) = 1$$



# Theorem

$$\alpha = 2$$

- Permits finding  $O(\log^2 n)$  intrinsic length paths
- Using local routing : Always move to the neighbor nearest to the destination

# Proof (optional)

- Main idea:
- In  $O(\log n)$  steps, the extrinsic distance is halved
  - Let us call this one phase
- In  $O(\log n)$  phases, the distance will be 1
- So, we need to show the first claim: one phase lasts  $O(\log n)$  steps

# One Phase (cover half the distance) lasts $\log n$ steps

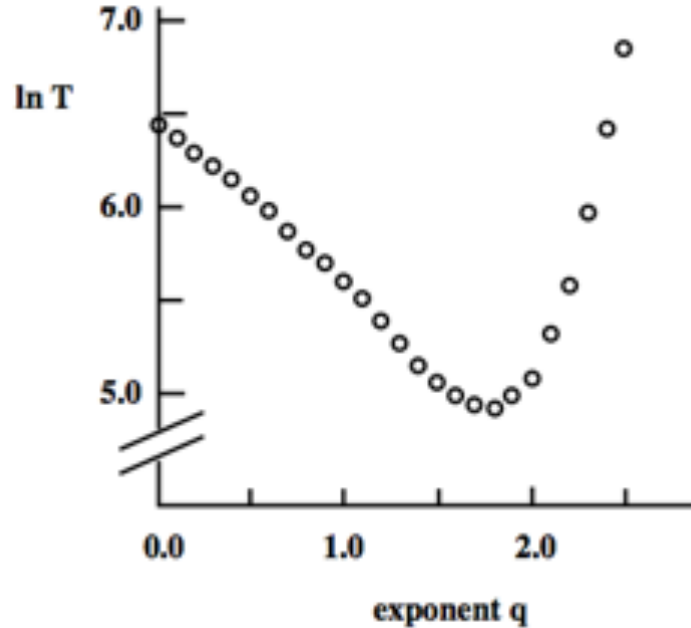
- Suppose distance from  $s$  to  $t$  is  $d$
- take ball  $B$  of radius  $d/2$  around  $t$
- There are about  $\Theta(d^2)$  nodes in this area
- The probability that a long link hits  $B$  is

$$\frac{1}{\Theta(\log n)} \sum_{v \in B} d(s, v)^{-2} \geq \Theta \left( \frac{1}{\log n} d^2 d^{-2} \right) = \Theta \left( \frac{1}{\log n} \right)$$

- One phase lasts  $\log n$  steps
- Thus, the expected number of steps before we find a link into B is  $\log n$ .
- And there are  $\log n$  such phases
- Therefore, this method finds a path of  $O(\log^2 n)$  steps

# Other exponents

- $< 2$  : more like uniform random
- $> 2$  : Shorter links, almost same as basic grid..



# Generality

- Search is a very general problem
- Search for an item, search for a path, search for a set, search for a configuration
- Decentralized: Operation under small amount of information. (local, easy to distribute)

# Small worlds in other networks

- Brain neuron networks
- Telephone call graphs
- Voter network
- Social influence networks ...
  
- Applications:
- Peer to peer networks
- Mechanisms for fast spread of information in social networks
- Routing table construction