Power law networks

Social and Technological Networks

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Degree distribution

- A way of characterizing networks
- More complex than single numbers
- Many standard networks are known to have "standard" degree distributions
- Gives ways to incorporate notions of "popularity" and understand them

Degree distribution

- As a function of *k*, what fraction of pages in the network have *k* links?
- A histogram

- What does it look like in a random graph?

Degree distribution of a random graph

- Probability that a node has degree k is:
  - Given by binomial distribution:

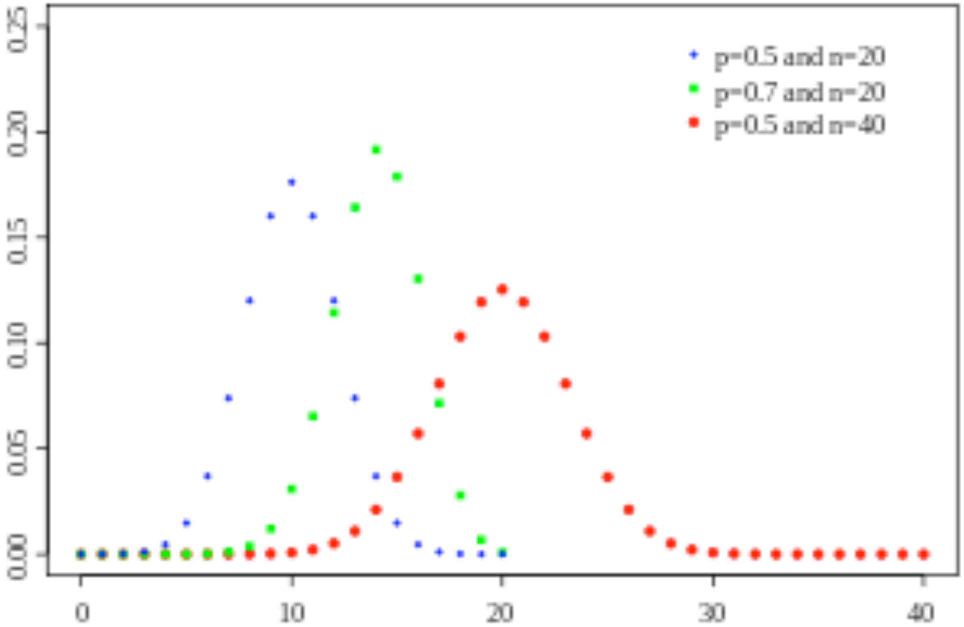
$$\binom{n-1}{k} p^k (1-p)^{n-1-k}$$

Possible sets of k edges

Probability that all k are chosen

Probability that others are not chosen

Degree distribution in a random graph



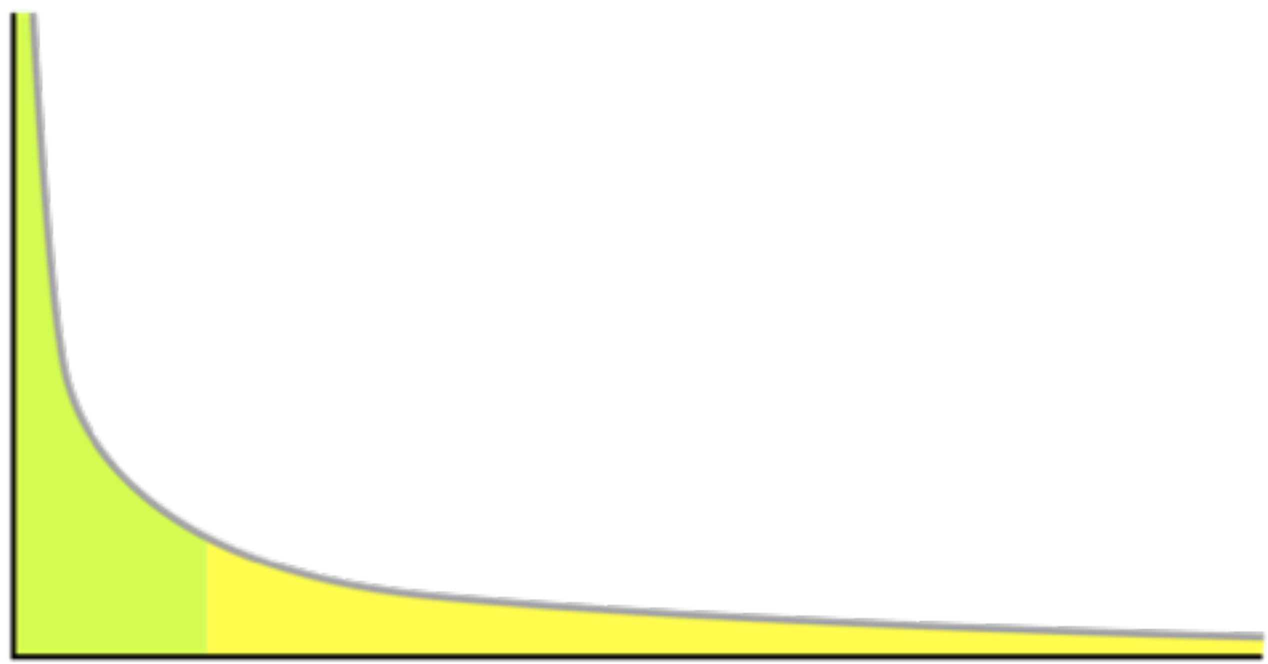
- Probabilities fall off really fast away from the peak
  - Exponentially fast with k
  - Very low and high degree are very very unlikely

Degree distribution in www

- Suppose we take a real network like the world wide web, and compute degree distribution. What does that look like?
- Let's try.

Degree distribution in www

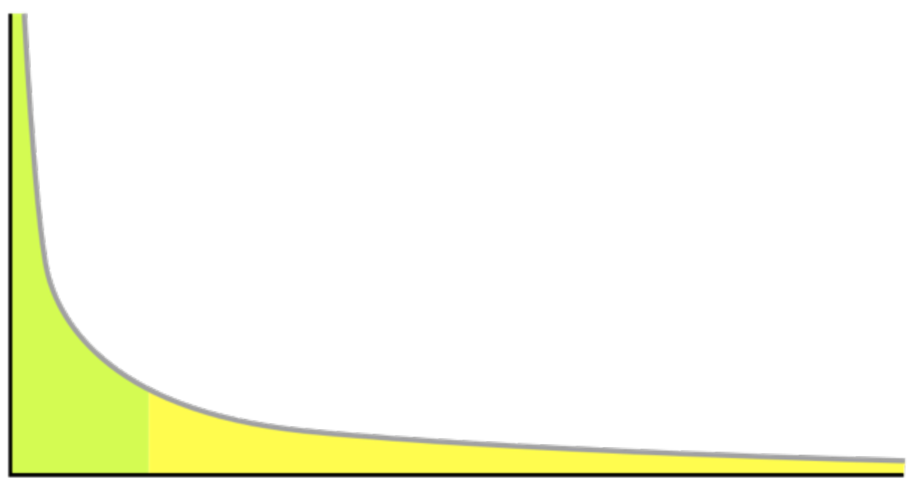
- For www snapshots, degree distribution follows approximately $\frac{1}{k^2}$



Power law networks

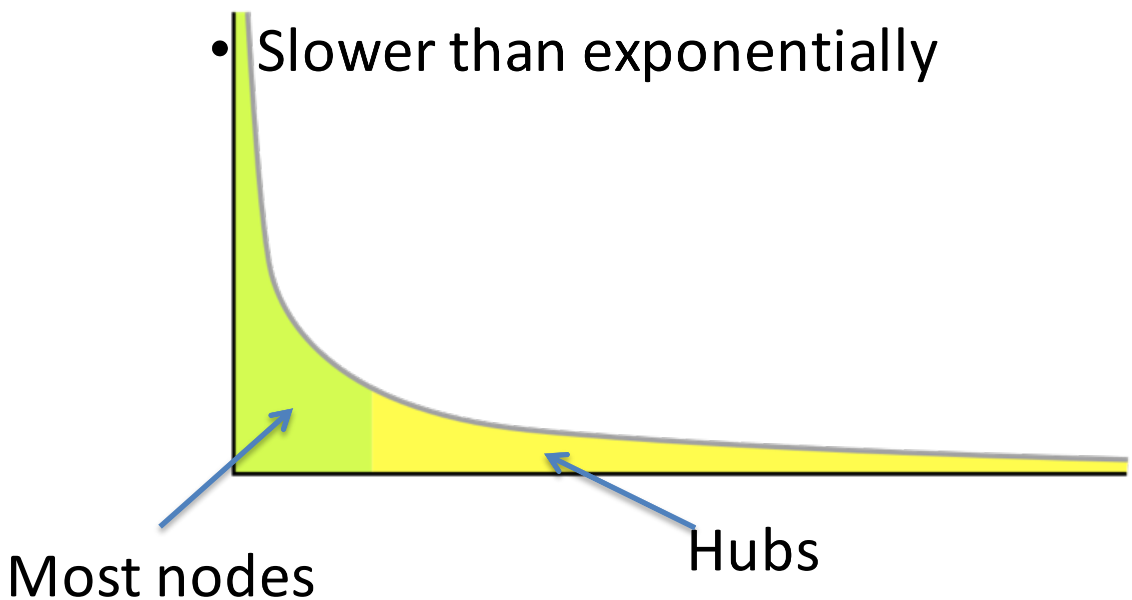
- With degree distribution $\frac{1}{k^{\alpha}}$

- For some constant α



What do power law networks mean

- Most nodes have a low degree
- There are several hubs with high degree
  - Heavy tail
  - Probability drops polynomially
    - Slower than exponentially



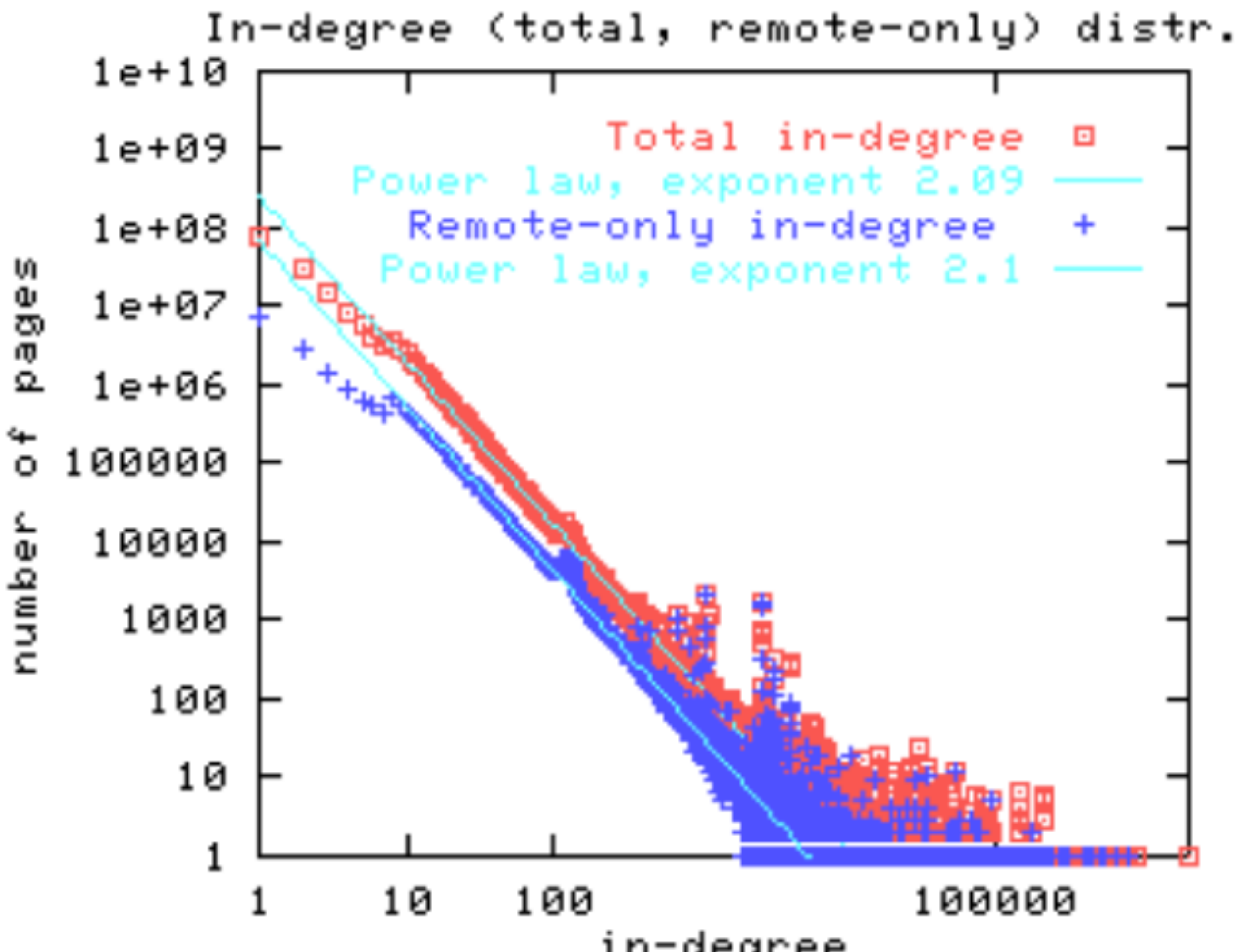
Hubs in power law networks

- Highly connected people/entities
- Critical in information dissemination
- Causes the network to have small diameter
- Examples
  - www, internet..
  - Social networks
  - Collaboration networks

Log log plots

- On ipython notebook

Log log plots for power law are nice and straight



Be careful with log log plots

- The "straight" part needs to extend quite a few orders of magnitude for the pattern to be significant
- Fitting the straight line to determine the right coefficient alpha is not trivial due to non-linear nature of data
- Beware: log-normal distributions can look similar to power law.

Mean degree in a power law distribution

- The mean is finite iff $\alpha > 2$
  - (On an infinite graph)

- On the www α is slightly larger than 2

Model of power law networks

- We want a model that can be used to create power law networks
- Preferably one that mimics creation of actual power law networks like www
  - Gives us some idea of how these networks were created

Preferential attachment mechanism

- Idea: older and established (popular) sites are likely to have more links to them (yahoo, google…)
- So how about: When a new page arrives, it links to older pages in proportion to their popularity
- When a new link is created on a new page, randomly to older pages with probability of hitting a page x proportional to current popularity of x (number of links to x)

Preferential attachment model

- Takes a parameter p in [0,1]
- On a new page, create k links as follows:
- When creating a new link:
- With probability p
  - Assign it with preferential attachment mechanism
- With probability 1-p
  - Assign it with uniform random probability to any existing page

Preferential attachment model

- Takes into consideration that popularity is not the only force behind link creation.
- The randomly assigned links model other reasons for link creation.
- Can be proven to produce power law. see [Kempe lecture notes, 2011]
- Produces same exponent as www for p~0.9
- Let’s see in the data

Power law often appears in other places

- Popularity of books
- Popularity of people, songs, ….

- Preferential attachment & power law are often a signature of artificial selection and popularity

Other reasons for power law

- Optimization:
  - Power law found in linguistics (word lengths): most frequent words are short
    - Mandelbrot, Zipf : emerges from need for efficient communication
- Random processes:
  - Press space with probability p, else press a random letter key
  - This will produce a power law distribution of word lengths