Social and Technological Networks Edinburgh, 2018

Exercises 4. Growth, explansion and doubling dim - solutions.

Rik Sarkar *Exercises*

Q 1. *Consider the unweighted grid graph with the intrinsic metric. Is this metric space the same as the two dimensional plane $\mathbb{R}^2$ with the L_1 metric?*

Answer. No. Because the grid distances are defined only between vertices of the graph (a countable set) and can have integer values. The general L_1 metric in the plane is defined between any two points in the plane (an uncountable set), and can have real values.

Q 2. *What do balls in the plane $\mathbb{R}^2$ look like, when considered in L_1, L_2 and L_∞ metrics? [Hint: Take a grid, and draw l_1 and L_∞ balls.]*

Answer. See wikipedia :`https://en.wikipedia.org/wiki/Unit_sphere`

Q 3. *Check that in a balanced binary tree, a ball of radius r cannot be covered by a constant number of balls of radius $r/2$.*

Q 4. *What are the Growth, Doubling dimension and Expansion of following types of graphs:*

- *The $1-D$ chain.*
- *The $2-D$ grid.*
- *The balanced binary tree*

Q 5. *Suppose we have a set of wireless nodes arranged in a finite grid with sides $1/2$ and we consider the unit disk graph. Now suppose that we go an adding nodes at random and updating the UDG, how do the growth and doubling dimension change? In particular, suppose we have added nodes such that any disk of radius $1/2$ has $\Theta(k)$ nodes. What can you say about the growth and doubling dimension?*

Answer. Doubling dimension remains a constant. Growth increases with k: a ball of radius r in this network will have $\Theta(kr^2)$ nodes. [Note that in this case k is a variable parameter – we are interested in the changing behaviour of growth with change in k.]

Q 6. *A graph is said to have bounded growth (sometimes called polynomial bounded growth), if the growth of metric balls is bounded by $O(r^\rho)$ for some constant ρ. Prove that a graph with bounded growth has bounded doubling dimension. [Also prove that bounded doubling dimension does not imply bounded growth.]*

Answer. That bounded doubling dimension does not imply bounded growth is implied by the example above.

For the other claim, suppose that a graph G has bounded growth ρ such that $|B_r(p)| = \Theta(r^\rho)$. Consider a ball $B_{2r}(p)$, we use a greedy algorithm to select balls of radius r to cover it. In particular, we select a node q in $B_{2r}(p)$ that is not yet covered, and cover all nodes in $B_r(q)$. Iterate until all nodes are covered. Now we bound how many balls are selected (denote this set as Q). To see the bound, we take the selected nodes $q \in Q$ and the balls $B_{r/2}(q)$. They do not overlap as any two nodes in Q are of distance at least r away. Thus, $|Q| \leq |B_{2r}(p)|/\min(|B_{r/2}(q)|) = O(\frac{(2r)^\rho}{(r/2)^\rho}) = O(4^\rho)$. This implies a doubling dimension of $O(2\rho)$.

Q 7. *Suppose we generalize the definition of doubling dimension to c-multiplicative dimension. That is, a metric space has bounded c-multiplicative dimension if any ball of radius r can be covered by a constant number of balls of radius r/c. Show that bounded doubling dimension implies bounded c-multiplicative dimension, and vice versa.*

Answer. Let us write doubling dimension as η. A ball of radius r can be covered with 2^η balls of radius $r/2$. We recursively cover each such $r/2$ ball with balls of half their radius, until the size of balls used falls below r/c. The resultant number of balls is $2^{\eta k}$, where $k = \lceil \lg c \rceil$. This is equivalent to $O(c^\eta)$ balls of radius r/c covering a ball of radius r.

[These last two proofs are adapted from: `http://homepages.inf.ed.ac.uk/rsarkar/papers/spatial-routing-journal.pdf`.]