Small world networks

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Course

- Please make sure to read lecture notes and work out the exercises.
- Future lectures may depend partially on them

Small world

Milgram’s Experiment

- Take people from random locations in USA
- Ask them to deliver a letter to a random person in Massachusetts
- A person can only forward the letter to someone you know
- Question: How many hops do the letters take tto get to destination?

Results

- Out of 296 letters, only 64 completed
- Number of hops varied between 2 and 10
- Mean number of hops 6
- There were a few people that were the last hop in most cases

Discussion of experiment

Discussion of experiment

- Short paths exist between pairs (small diameter)
- More surprisingly, people find these short paths
- Without knowing the entire network
    - *Decentralized search*
    - Analogous to routing without a routing table
- People use a “greedy” strategy
    - Forward to the friend nearest to the destination

Recent results

- Milgrams results reproduced on better data
- Use online data (Livejournal, facebook)
    - Containing approximate locations
- Simulate the process of forwarding letters
- Results similar to original experiment
    - Relatively short diameter, successful decentralized search

In popular culture

- Erdos distance
- Kevin bacon distance

Definition of Small worlds

- Small diameter
- Large clustering coefficient
  - Related to homophily — similar people connect to each-other
  - “Similar”: close in some coordinate value
- Supports decentralized search
  - People find short paths without knowing the entire network
- (Usually) High expansion

Model 1: Watts and Strogatz

Nature 1998

- Parameters

$$n, k, \beta \qquad n > k > \ln n$$

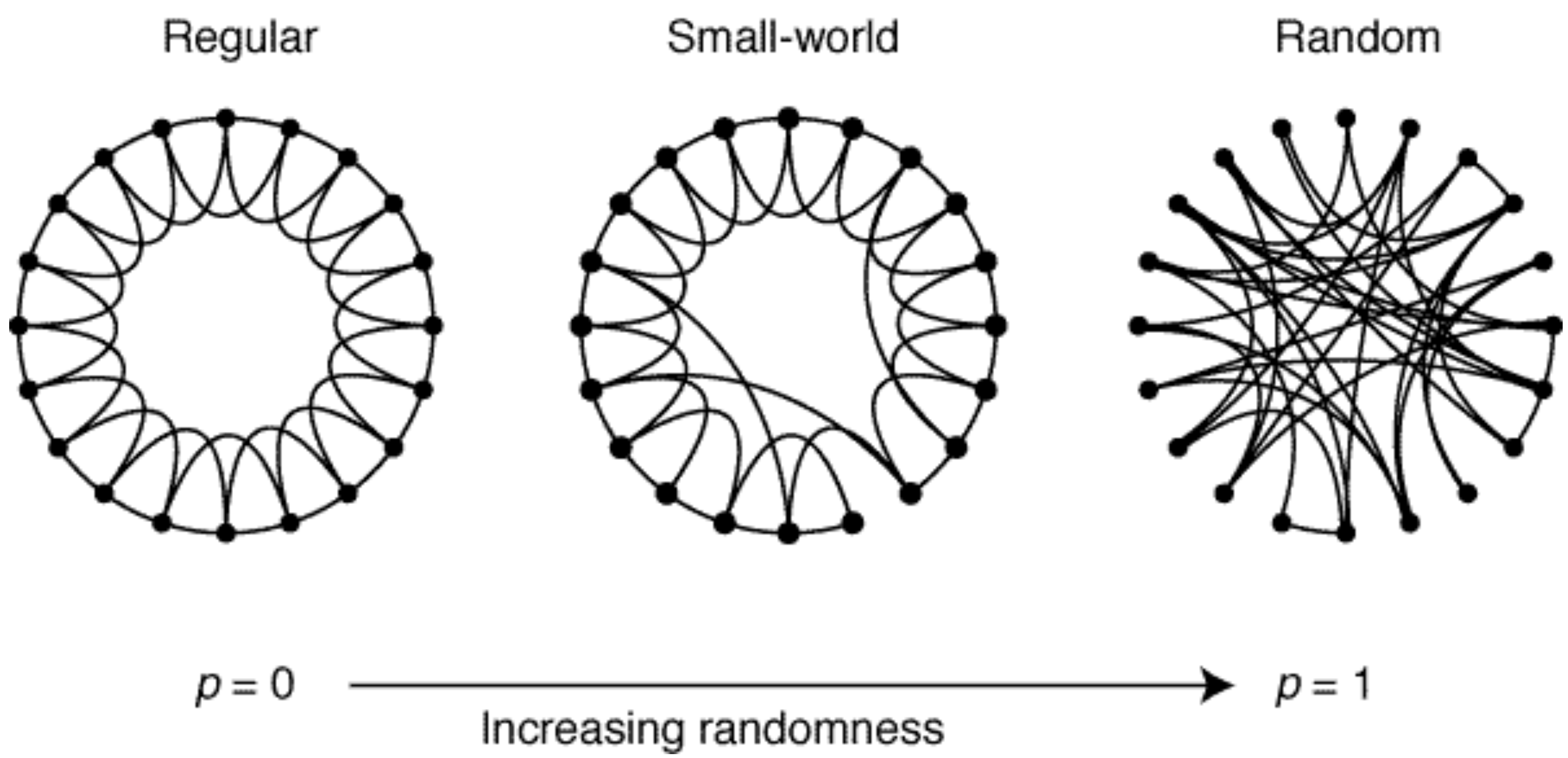
  - Often k is taken to be a constant with the idea that people cannot have infinitely large friend-circles

- Put nodes in a ring of size n

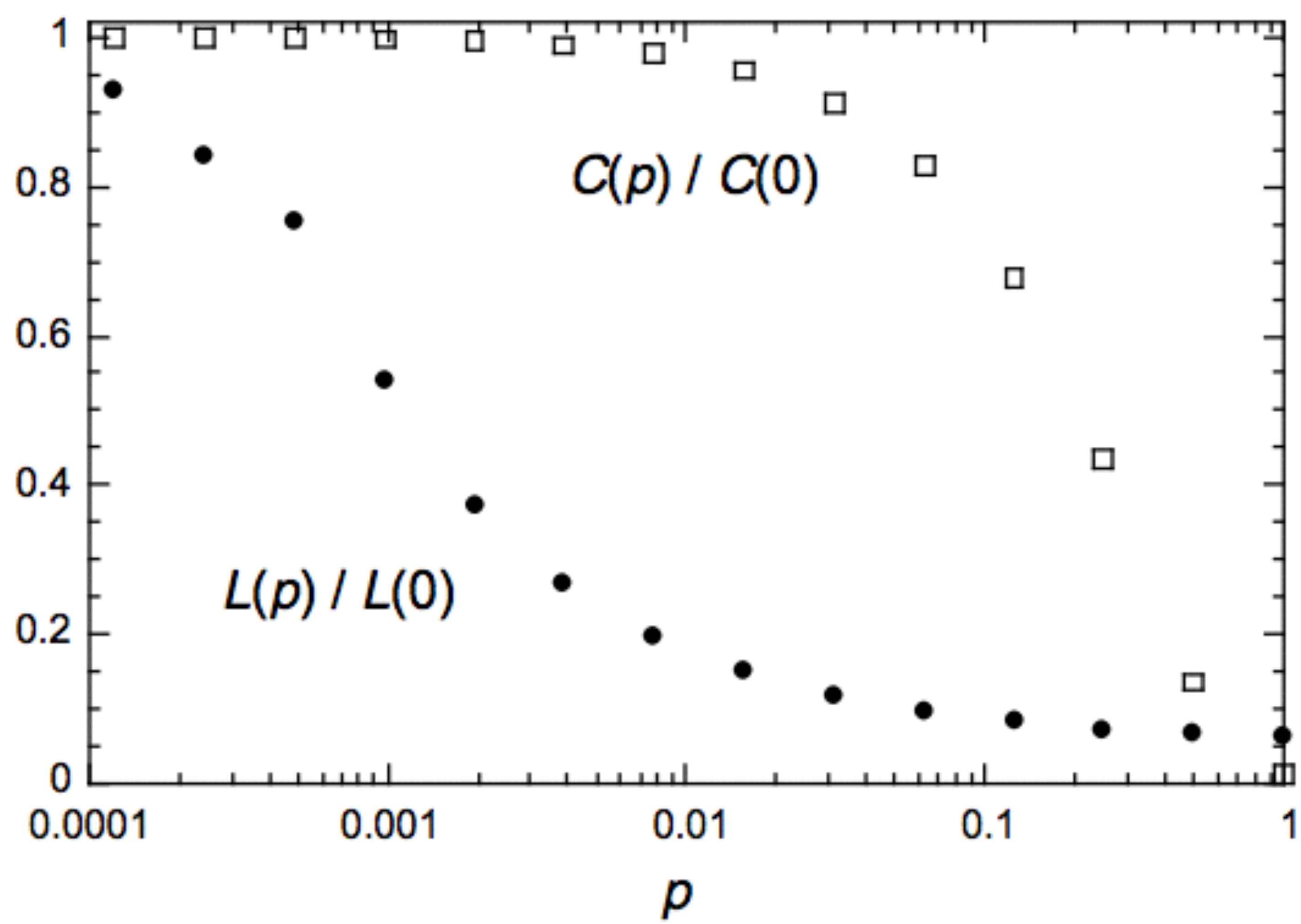
- Connect each to k/2 neighbors on each side

- With probability β rewire each edge of a vertex to a random vertex

Small world



Small world



Properties

- Average clustering coefficient per vertex bounded away from zero
- Connected: sufficient random edges + regular edges
- Short diameter

Watts-strogatz model does not explain milgram’s experiment

Watts-strogatz model does not explain milgram’s experiment

- Milgram’s experiment was on 2D plane
- Watts strogatz does not support decentralized search

Random link model

- We want to show that Watts-Strogatz model idea has good clustering coefficient and diameter but does not allow decentralized search
- Let us simplify the model slightly without changing anything important:
  - We put nodes on a plane or line or ring
    - From each node connect nodes that are “close” in the plane or line distance e.g. distance h unit or less
    - Then from each node add some edges to random nodes in the network

Clustering in Random link model

- First, consider the “short” links to nearby nodes
- This ensures that clustering coefficient is not too low.
- Idea: On a line or ring
  - All nodes to the right of x are connected to each other
  - Same on the left
  - Thus, sufficient number of edges exist between neighbors of x to have a high clustering coefficient

Diameter in random link model

- Without random links, diameter is poly(n)
    - e.g. n/h in line or ring
- With the random links, diameter is small
    - Idea: each random link can reduce the diameter a lot
    - Proof: Omitted for now

Decentralized search in random link networks

- Decentralized search does not work to produce short paths
- Let us consider 2D (n x n area):
  - We want to show that if every node works only on its local information (edges it has)
  - Then there is no algorithm that delivers the message in less than poly(n) messages.

Decentralized search in random link networks

- Specifically, we are going to show that if nodes are far in the plane,
    - It can take $\sim n^{2/3}$ steps
- Take s and t at distance $> n^{2/3}$
    - If we use only local edges, it will take $> n^{2/3}$ hops

Decentralized search in random link networks

- Take ball B of radius around $n^{2/3}$ t
  - There are $O(n^{2/3})^2$ nodes in B
- Since we are already at distance $n^{2/3}$
  - A long link can help only if it falls inside B
  - Otherwise we take a step along a short link
- What is the probability that a random link from hits B?
  - This is ~ $O((n^{2/3})^2/n^2) = O(n^{-2/3})$
- The expected number of steps before getting a useful long link is : $\Omega(n^{2/3})$

Decentralized search

- Therefore long links are not really useful in reaching t
- The number of steps is poly(n).

Model 2 : Kleinberg's model

STOC 2000, Nature 2000, ICM 2006

- Idea: Long links are not helping much
  - Getting closer to the destination does not increase the chances of getting a long link close to destination.
- Make the probability of a long link sensitive to the distance
  - Nearby nodes are more likely to have a long link

Model 2 : Kleinberg's model

- Suppose $d(u, v)$ is the distance between nodes u and v in the plane
- Then u connects its long link to v
  - with probability $\propto \frac{1}{d(u,v)^{\alpha}}$

Kleinberg’s model

- Links to nearby nodes are more likely
    - A node knows more people locally
    - With increasing distance, it knowns fewer and fewer people
    - At the largest scale it knows only a handful
    - More representative of how people have their contacts spread
- We want to show that the model permits short paths to be found

The proportionality constant

$$\Pr[(u,v)] = \tfrac{1}{\gamma}\tfrac{1}{d(u,v)^{\alpha}}$$

$$\alpha = 2 \Rightarrow \gamma = \Theta(\ln n)$$

- Sketch of proof: Take rings of thickness 1 at distances 1,2,3…
- The number of nodes at distance d ~ Θ(d)
- Thus from any node: $$\frac{1}{\gamma}\sum_{d=1}^{n} d^{-2}\Theta(d) = 1$$

$$\frac{1}{\gamma}\Theta\left(\sum_{d=1}^{n}\frac{1}{d}\right) = 1$$

$$\Rightarrow \frac{1}{\gamma}\Theta(\ln n) = 1$$

Theorem

$\alpha = 2$

- Permits finding $O(\log^2 n)$ length paths
- Using local routing : Always move to the neighbor nearest to the destination

Theorem

- Main idea:
  - In O(log n) steps, the distance is halved
    - Let us call this one phase
  - In O(log n) phases, the distance will be 1
  - So, we need to show the first claim: one phase lasts O(log n) steps

One Phase lasts log n steps

- Suppose distance from s to t is d
- take ball B of radius d/2 around t
- There are about $\Theta(d^2)$ nodes in this area
- The probability that a long link hits B is

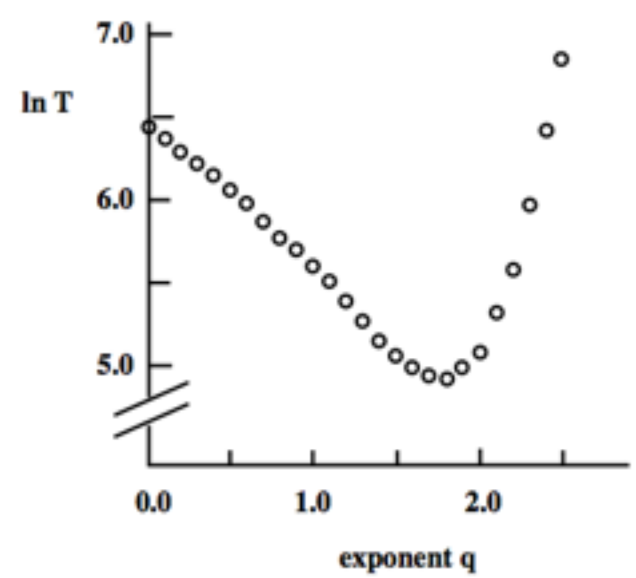
$$\frac{1}{\Theta(\log n)} \sum_{v \in B} d(s,v)^{-2} \geq \Theta\left(\frac{1}{\log n} d^2 d^{-2}\right) = \Theta\left(\frac{1}{\log n}\right)$$

One phase lasts log n steps

- Thus, the expected number of steps before we find a link into B is log n.
- And there are log n such phases
- Therefore, this method finds a path of $\log^2 n$ steps

Other exponents

- < 2 : more like uniform random
- > 2 : Shorter links, almost same as basic grid..



Other types of distances

- Manhattan distance, road network distance, time of travel
  - Graph distance, shortest path
- Size of the smallest ball containing both
- Distance in professional area
  - Computer science — maths—physics
- Social foci: Clubs/communities/teams
  - Distance: Size of the smallest foci containing both
  - …

Generality

- Search is a very general problem
  - Search for an item, search for a path, search for a set, search for a configuration
- Decentralized: Operation under small amount of information. (local, easy to distribute)

Small worlds in other networks

- Brain neuron networks
- Telephone call graphs
- Voter network
- Social influence networks …

- Peer to peer networks
- Mechanisms for fast spread of information in social networks
- Routing table construction

Next

- This ends the first part, random graphs and random graph based models
- Please read lecture notes and do exercises. Read other “reading” material.
- Next we will study world wide web
  - Structure of the web
  - Ranking algorithm: Pagerank and others
- Please revise your Linear algebra
  - Matrices, basic operation on matrices
  - Definitions and meaning of eigen vectors and eigen values.