Recap: Dynamical Systems

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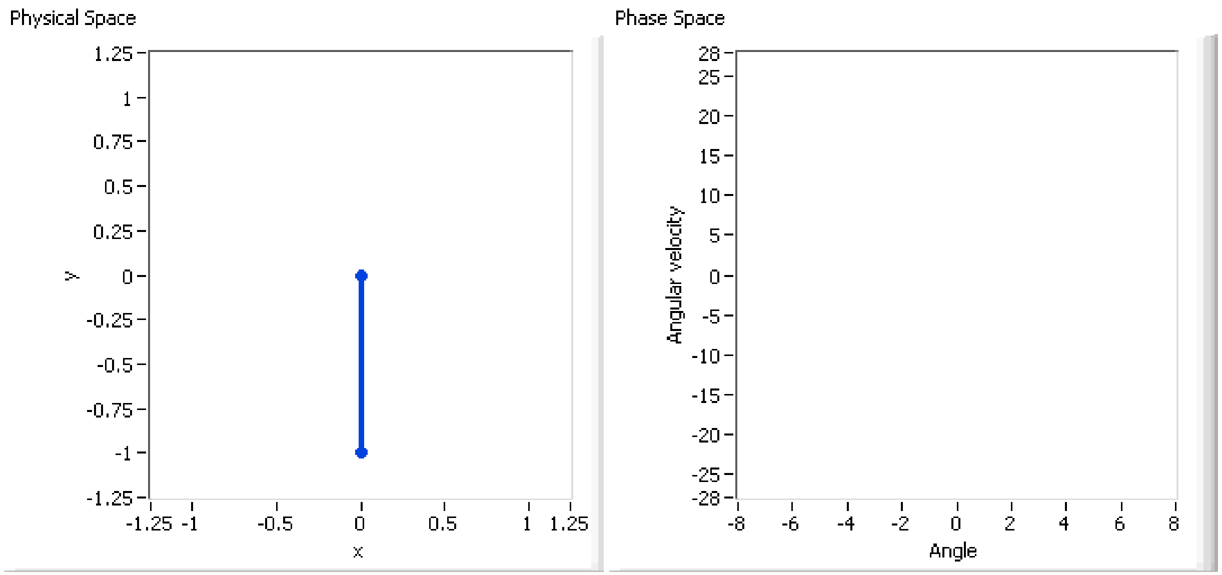
What can we Infer from Dynamics Models?

- Long-term dynamic behaviour
  - Stability: Will the dynamics converge? Will it come to rest?
  - Transient Response: How much will the state fluctuate in response to perturbations?
  - Given a certain family of control strategies, can this system be stabilized?

- Global Properties
  - Given that most of these equations are nonlinear, what kinds of phase space trajectories are possible?
  - What is the local structure along the various paths?

Example: Pendulum Phase Space



- Phase space is organized into families (open sets) of trajectories
- The phase space curves are parameterized by increasing energy

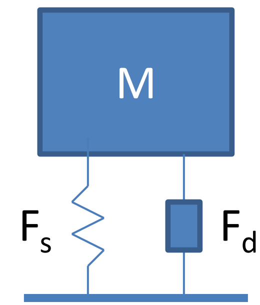
How do we describe more complex robots, analytically?

Linear Time Invariant (LTI) Systems

- Consider the simple spring-mass-damper system:
- The force applied by the spring is $F_s = -kz(t)$
- Correspondingly, for the damper: $F_d = \gamma \dot{z}(t)$
- The combined equation of motion of the mass becomes:

$$x\ddot{z}(t) = -\gamma \dot{z}(t) - kz(t)$$

- One could also express this in state space form:



$$x(t) = [x_1(t), x_2(t)]' = [z(t), \dot{z}(t)]'$$

$$\dot{x}(t) = \begin{pmatrix} \dot{z}(t) \\ \ddot{z}(t) \end{pmatrix} = \begin{pmatrix} x_2(t) \\ -\frac{1}{m}(\gamma x_2(t)) + kx_1(t) \end{pmatrix}$$

Linear ODE ⟵ $$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & \frac{\gamma}{m} \end{pmatrix} x(t) = Ax(t)$$

Solution of a Linear ODE

$\dot{x} = kx$, $x \in \mathbb{R}$

For initial condition $\phi(0) = x_0$, the solution is $\phi(t) = e^{kt}x_0$

i.e., time evolution of state is given by operator $g^t = e^{kt}$, with velocity $\mathbf{v} = kt$

This type of "exponential term" is a feature of all linear dynamical systems

The multivariate case: $x(t) = e^{A(t-t_0)}x_0$

$$\begin{aligned} \boxed{e^{A(t-t_0)}} &= \sum_{i=0}^{\infty} \frac{A^i (t-t_0)^i}{i!} \\ &= I_{n\times n} + A(t-t_0) + \frac{A^2(t-t_0)^2}{2!} + \dots \end{aligned}$$

This is state transition matrix **ϕ(t) :**
In linear algebra, there are
numerous ways to compute this…

Example

Determine the matrix exponential, and hence the state transition matrix, and the homogeneous response to the initial conditions $x_1(0) = 2$, $x_2(0) = 3$ of the system with state equations:

$$\begin{aligned}\dot{x}_1 &= -2x_1 + u \\ \dot{x}_2 &= x_1 - x_2.\end{aligned}$$

The system matrix is

$$\mathbf{A} = \begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix}.$$

Example, contd.

$$
\begin{aligned}
\mathbf{\Phi}(t) &= e^{\mathbf{A}t} \\
&= \left(\mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2t^2}{2!} + \frac{\mathbf{A}^3t^3}{3!} + \ldots + \frac{\mathbf{A}^kt^k}{k!} + \ldots\right) \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix} t + \begin{bmatrix} 4 & 0 \\ -3 & 1 \end{bmatrix} \frac{t^2}{2!} \\
&\qquad + \begin{bmatrix} -8 & 0 \\ 7 & -1 \end{bmatrix} \frac{t^3}{3!} + \ldots \\
&= \begin{bmatrix} 1 - 2t + \frac{4t^2}{2!} - \frac{8t^3}{3!} + \ldots & 0 \\ 0 + t - \frac{3t^2}{2!} + \frac{7t^3}{3!} + \ldots & 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \ldots \end{bmatrix}.
\end{aligned}
$$

$$
\mathbf{\Phi}(t) = \begin{bmatrix} e^{-2t} & 0 \\ e^{-t} - e^{-2t} & e^{-t} \end{bmatrix}
$$

Example, contd.

$$\mathbf{x}_h(t) = \mathbf{\Phi}(t)\mathbf{x}(0)$$

$$\begin{aligned}
x_1(t) &= x_1(0)e^{-2t} \\
x_2(t) &= x_1(0)\left(e^{-t} - e^{-2t}\right) + x_2(0)e^{-t}.
\end{aligned}$$

$$\begin{aligned}
x_1(t) &= 2e^{-2t} \\
x_2(t) &= 2\left(e^{-t} - e^{-2t}\right) + 3e^{-t} \\
&= 5e^{-t} - 2e^{-2t}.
\end{aligned}$$

Basic Notion: Stability

- Simple question:
  Given the system, $\dot{x}(t) = Ax(t)$
  where in phase space, $(x, \dot{x})$, will it come to rest?

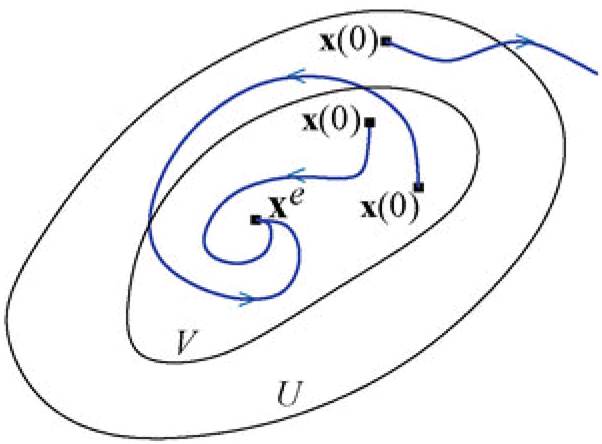
Any guesses?
Think about solution in previous slide…

- This point is called the equilibrium point
  - If initialized there, dynamics will not take it away
  - If perturbed, system will eventually return and stay there

Stability

An equilibrium position $x = 0$ is *stable* (in Lyapunov's sense) if given $\epsilon > 0$, $\exists\delta > 0$ (not dependent on t), s.t. $\forall x_0, |x_0| < \delta$ the solution satisfies $|\phi(t)| < \epsilon$, $\forall t > 0$

Asymptotic stability: Lyapunov stabile and $\lim_{t\to+\infty}\phi(t) = 0$



Stability for an LTI System, $\dot{x}(t) = Ax(t)$

Unforced (homogeneous) response: $x_i(t) = \sum_{j=1}^{n} m_{ij} e^{\lambda_j t}$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & \dots & m_{1n} \\ m_{21} & m_{22} & \dots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & \dots & m_{nn} \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ \vdots \\ e^{\lambda_n t} \end{bmatrix}$$

$$\mathbf{x_h}(t) = \mathbf{M} \begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ \vdots \\ e^{\lambda_n t} \end{bmatrix}$$

Stability for an LTI System

If you differentiate the homogeneous response, $\frac{dx_i}{dt} = \sum_{j=1}^{n} \lambda_j m_{ij} e^{\lambda_j t}$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 m_{11} & \lambda_2 m_{12} & \dots & \lambda_n m_{1n} \\ \lambda_1 m_{21} & \lambda_2 m_{22} & \dots & \lambda_2 m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1 m_{n1} & \lambda_2 m_{n2} & \dots & \lambda_n m_{nn} \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ \vdots \\ e^{\lambda_n t} \end{bmatrix}.$$

The system being considered is $\dot{x}(t) = Ax(t)$, so:

$$\begin{bmatrix} \lambda_1 m_{11} & \lambda_2 m_{12} & \dots & \lambda_n m_{1n} \\ \lambda_1 m_{21} & \lambda_2 m_{22} & \dots & \lambda_2 m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1 m_{n1} & \lambda_2 m_{n2} & \dots & \lambda_n m_{nn} \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ \vdots \\ e^{\lambda_n t} \end{bmatrix} = \mathbf{A} \begin{bmatrix} m_{11} & m_{12} & \dots & m_{1n} \\ m_{21} & m_{22} & \dots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & \dots & m_{nn} \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ \vdots \\ e^{\lambda_n t} \end{bmatrix}$$

LTI Stability, in algebraic equations

- The above equation leads to an eigenvalue problem:

$$\lambda_i \mathbf{m}_i = \mathbf{A}\mathbf{m}_i \qquad\qquad i = 1, 2, \ldots, n.$$

$$\left[\lambda_i \mathbf{I} - \mathbf{A}\right] \mathbf{m}_i = 0$$

- For this to have nontrivial solutions:

Characteristic eqn.

$$\Delta(\lambda_i) = \det\left[\lambda_i \mathbf{I} - \mathbf{A}\right] = 0.$$

$$\lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \ldots + a_1\lambda + a_0 = 0$$

$$(\lambda - \lambda_1)(\lambda - \lambda_2)\ldots(\lambda - \lambda_n) = 0.$$

Stability: LTI System, $\dot{x}(t) = Ax(t)$

Theorem. *Let* λ_i, $i \in \{1, 2, \ldots, n\}$ *denote the eigenvalues of* A. *Let* $re(\lambda_i)$ *denote the real part of* λ_i. *Then the following holds:*

1. $x_e = 0$ *is stable if and only if* $re(\lambda_i) \leq 0$, $\forall i$
2. $x_e = 0$ *is asymptotically stable if and only if* $re(\lambda_i) < 0$, $\forall i$
3. $x_e = 0$ *is unstable if and only if* $re(\lambda_i) > 0$, *for some* i

For the spring-mass-damper example, the eigenvalues are:

$$\frac{\gamma \pm \sqrt{\gamma^2 - 4km}}{2m}$$

With positive damping, we get asymptotic stability

Exercise

Can you visualize (i.e., draw the curve vs. time) state variables for the case of asymptotic stability, instability and the borderline in between?

But, Most Robots are Non-linear Systems…

- One way to analyze such systems is through local linearization
  - Determine a state of interest, fit linear model around it

- Consider a dynamics model:

$$\frac{d}{dt}\mathrm{x}_n(t) = \mathcal{F}(\mathrm{x}_n(t), \mathrm{f}_n(t))$$

Velocity

Nonlinear equations
of motion

Applied forces
e.g., control actions

Taylor Series Expansion

$$\frac{d}{dt}\mathrm{x}_n + \frac{d}{dt}\Delta\mathrm{x} = \mathcal{F}(\mathrm{x}_n + \Delta\mathrm{x}, \mathrm{f}_n + \Delta\mathrm{f})$$

$$= \mathcal{F}(\mathrm{x}_n, \mathrm{f}_n) + \left(\frac{\partial\mathcal{F}}{\partial\mathrm{x}}\right)_{\big|^{\mathrm{x}_n(t)}_{\mathrm{f}_n(t)}} \Delta\mathrm{x} + \left(\frac{\partial\mathcal{F}}{\partial\mathrm{f}}\right)_{\big|^{\mathrm{x}_n(t)}_{\mathrm{f}_n(t)}} \Delta\,\mathrm{f} + \text{higher–order terms}$$

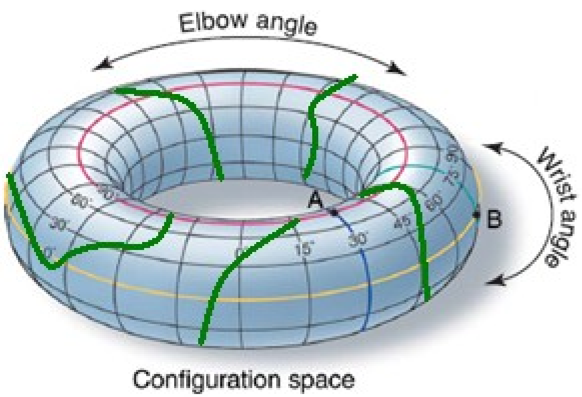
$$\frac{d}{dt}\Delta\mathrm{x}(t) = \left(\frac{\partial\mathcal{F}}{\partial\mathrm{x}}\right)_{\big|^{\mathrm{x}_n(t)}_{\mathrm{f}_n(t)}} \Delta\mathrm{x}(t) + \left(\frac{\partial\mathcal{F}}{\partial\mathrm{f}}\right)_{\big|^{\mathrm{x}_n(t)}_{\mathrm{f}_n(t)}} \Delta\mathrm{f}(t)$$

$$\frac{d}{dt}\Delta\mathrm{x}(t) = \mathrm{A}\Delta\mathrm{x}(t) + \mathrm{B}\Delta\mathrm{u}(t), \quad \Delta\mathrm{x}(t_0) = \mathrm{x}(t_0) - \mathrm{x}_n(t_0)$$

-Local linear system: do linear analysis
OR, find a Lyapunov function directly!

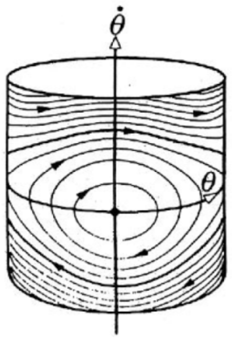
Domain for the Dynamics

- Sometimes, the dynamics evolves on a surface
  - Configuration space with interesting structure (e.g., space of shapes of a distributed robot)
  - Constraint manifold (set of c-space points subject to constraint)



In these instances,
a more abstract description
is often helpful…

Dynamics as Phase Flow



Phase space: M

Initial state: $x \in M$

State at time t: $g^t x$

Dynamics is a mapping from phase space to itself: $g^t : M \to M, \forall t \in \mathbb{R}$

g^t is a t-advance mapping with the property, $g^{t+s} = g^t g^s$

$\{g^t x, t \in \mathbb{R}\}$ describes a "phase curve", a subset of phase space

A family of t-advance mappings ($\forall x \in M$) constitutes a "phase flow"

Notion of Fixed Point (Equilbrium)

- There might be points at which the system tends not to move
- If the system is initialized at that point, it stays there forever
- In other words, the point maps back to itself:

$$g^t x = x, \forall t \in \mathbb{R}$$

- Many "controllers" used in robotics act to ensure that some desired point in phase space is a fixed point

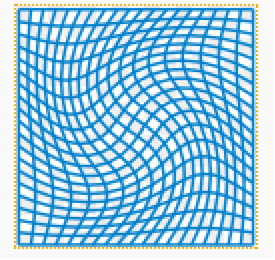
  e.g., what do you do when holding a glass of water in your hand ?

Useful Viewpoint: Dynamics as Diffeomorphism

The *phase flow* is a diffeomorphism – maps phase space points to other phase space points:

$$g : \mathbb{R} \times M \to M$$

$$g(t, x) = g^t x, t \in \mathbb{R}, x \in M$$



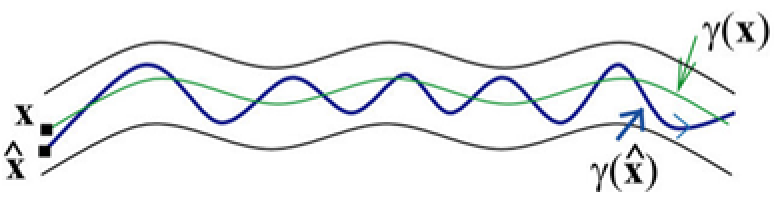
- g is a differentiable mapping
- $g^t : M \to M$ is a diffeomorphism for every $t \in \mathbb{R}$
- The family $\{g^t\}$ is a 1-parameter group of transformations of M

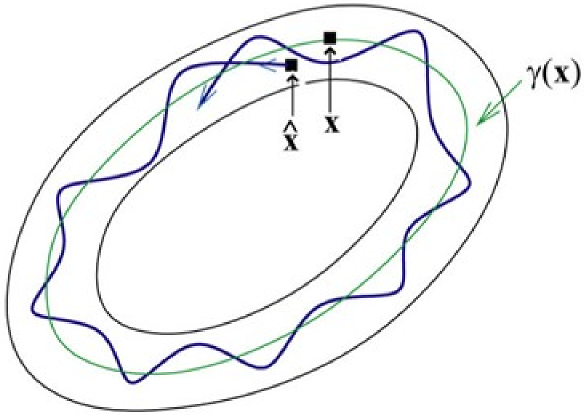
Can you visualize fixed points in these terms?

Orbital Stability

- From this viewpoint, stability doesn't have to always be about coming to rest at a point
  - could be defined in terms of staying in a subset, e.g., path

Definition. *An orbit $\gamma(x)$ is orbitally stable if for any $\epsilon > 0$, there is a neighbourhood V of x so that for all $\hat{x}$ in V, γx and $\gamma\hat{x}$ are ϵ-close.*
Loosely speaking, $|\gamma(x) - \gamma(\hat{x})| < \epsilon$ at all times.





Useful Concept: Vector Field

$(M, \{g^t\})$:
phase flow given by a 1-param group of diffeomorphisms on manifold M

Phase velocity of flow g^t at point $x \in M$ is a vector: $\mathbf{v}(x) = \frac{d}{dt}\Big|_{t=0} g^t x$

Aggregate phase velocity forms a vector field on phase space M

Points where this vector vanishes are "singular points"