Randomness and Computation 2018/19
Week 8 tutorial sheet (12-1pm, Tues 12th, Wed 13th March)

1. We are given an undirected graph $G = (V, E)$ where each $v \in V$ is associated with a set of $8r$ colours $S(v)$, for some $r \geq 1$.

   The $S(v)$ sets may overlap or in some cases be identical, or anything in between; however we have the guarantee that for every $v \in V$ and every $k \in S(v)$, there are at most r neighbours $u \in \mathsf{Nbd}(v)$ that also have $k \in S(u)$.

   We now show there is a proper (vertex) colouring which assigns a colour from $S(v)$ to each $v \in V$ such that for every $e = (u, v) \in E$, u and v get different colours.

   Consider the following procedure for colouring the vertices of G: for each vertex $v \in V$ we independently pick uniformly at random a colour $c \in S(v)$, and colour v with c. We will show that there is a strictly positive probability we end up with a proper colouring of G.

   For any pair $u, v \in V$ such that $(u, v) \in E$, $A_{u,v,c}$ denotes the event that both u and v are coloured with c. Observe that if $c \notin S(u)$ or $c \notin S(v)$, then $\Pr[A_{u,v,c}] = 0$. On the other hand, if $c \in S(u)$ and $c \in S(v)$, then $\Pr[A_{u,v,c}] = \frac{1}{8r}\frac{1}{8r} = \frac{1}{64r^2}$. Since for each vertex $v \in V$ and each colour $c \in S(v)$ there are at most r neighbours $u \in \mathsf{Nbd}(v)$ such that $c \in S(u)$, each event $A_{u,v,c}$ depends on at most $d(A_{u,v,c}) = 8r^2 + 8r^2 = 16r^2$ other events $A_{u',v',c'}$. Therefore, we have $\Pr[A_{u,v,c}] \cdot d(A_{u,v,c}) \leq \frac{1}{64r^2} \cdot 16r^2 = \frac{1}{4}$. Hence, by Lovasz Local Lemma we have that $\Pr[\bigcap_{u,v,c} \overline{A_{u,v,c}}] > 0$. Therefore, there must exist a colouring of G such that for any $(u, v) \in E$, u and v have different colours.

2. Let I_n be the number of isolated vertices in $G_{n,p}$. We can write $E[I_n] = n(1-p)^{n-1}$.If we have $p = p(n) = \frac{c\log(n)}{n}$ then we have

   $$n\left(1 - \frac{c\log(n)}{n}\right)^{n-1}$$

   which is approximately $ne^{-c\log(n)}$, and this itself will be $n \cdot n^{-c}$. If $c < 1$, then this quantity is n^{1-c} which tends to ∞ as n grows.

   We then need to consider the second moment, and we are lucky that for isolated vertices, the events for two different vertices (u, v) are *almost independent*, only depending on the status of the edge between u and v. We will have

   $$E[I_n^2] = \sum_{u\in V}\sum_{v\in V} \Pr[I_u I_v] \cdot 1 = \sum_{u\in V} \Pr[I_u] + 2\sum_{u,v\in V, u\neq v} \Pr[I_u I_v].$$

   We know $\Pr[I_u] = (1-p)^{n-1}$ for each u. Also, for each $u, v, u \neq v$, we know that both vertices are independent iff and only if $2(n-2)+1$ edges are omitted (the 1 is for the shared edge (u, v)), so we have

   $$E[I_n^2] = n(1-p)^{n-1} + n(n-1)(1-p)^{2n-3},$$

$$\mathrm{Var}[I_n] \;=\; n(1-p)^{n-1} + n(n-1)(1-p)^{2n-3} - n^2(1-p)^{2(n-1)}.$$

Now we use Theorem 6.7 which tells us that $\Pr[X=0] \leq \frac{\mathrm{Var}[X]}{\mathrm{E}[X]^2}$ for any non-negative r.v X with bounded $\mathrm{E}[X]$. So for I_n, we have that

$$\begin{aligned}
\Pr[I_n = 0] &\leq \frac{n(1-p)^{n-1} + n(n-1)(1-p)^{2n-3} - n^2(1-p)^{2(n-1)}}{n^2(1-p)^{2(n-1)}}. \\
&= \frac{1}{n(1-p)^{n-1}} + \frac{n-1}{n(1-p)} - 1 \\
&< \frac{1}{n(1-p)^{n-1}} + \frac{1}{(1-p)} - 1 \\
&\sim \frac{n^c}{n} + \frac{n}{n - c\log(n)} - 1 \\
&\sim \frac{1}{n^{1-c}} - \frac{c\log(n)}{n - c\log(n)} \\
&< \frac{1}{n^{1-c}}.
\end{aligned}$$

We know $c < 1$ and hence we have $\frac{1}{n^{1-c}} < 1$ and also we can achieve $\frac{1}{n^{1-c}} < \epsilon$ by setting $n > \frac{1}{\epsilon^{1/(1-c)}}$, this power for ϵ being a fixed constant under the assumption $c < 1$.

3. We are asked to determine a simple expression for $\mathbf{P}^t[0,1]$, for $\mathbf{P}$ defined as

$$\mathbf{P} \;=\; \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}$$

Note that this Markov chain codes a 1-step process over $\{0,1\}$ where p is the probability that the state keeps its current value, and $1-p$ is the probability it flips. If we take the square of this matrix, the value of $M^[0,1]$ would be the probability of $0 \to\to 1$ over two steps, which is $2p(1-p)$. Carrying forward this analysis, $M^t[0,1]$ is the probability of an odd number of flips, taken over t steps of the process. This is $\sum_{i=1}^{\lfloor t/2 \rfloor - 1} (1-p)^{2i+1} p^{t-1-2i} \binom{t}{2i}$. Inside the summation, it is basically the sum of the odd terms of the binomial expansion.

We can analyse by looking at $(a+b)^t - (a-b)^t$, which expands as

$$\left[\sum_{i=0}^{t} \binom{t}{i} b^i a^{t-i}\right] - \left[\sum_{i=0}^{t} (-1)^i \binom{t}{i} b^i a^{t-i}\right].$$

Then the terms with even-i cancel out, and the terms with odd-i double, to give the value $2\left[\sum_{i'=0}^{\lfloor t/2 \rfloor} \binom{t}{2i'+1} b^{2i'+1} a^{t-2i'-1}\right]$.

Now return to our sum of terms, and note it matches the expression just-above for $a = p, b = 1-p$. We have

$$\frac{1-(2p-1)^t}{2} \;=\; \sum_{i=0}^{\lfloor t/2 \rfloor - 1} (1-p)^{2i+1} p^{t-1-2i} \binom{t}{2i+1}.$$

This checks out with direct expansions for $t = 1, 2, 3$ by using the matrix.

4. This is a worked example, to help improve understanding of how the chain operates. Our row and column sums are $\mathsf{r} = (2, 2, 4), \mathsf{c} = (2, 3, 3)$ for this example, and we have the following *start state* X:

$$\mathsf{X} = \begin{array}{ccc||c} \hline 2 & 0 & 0 & 2 \\ \hline 0 & 2 & 0 & 2 \\ \hline 0 & 1 & 3 & 4 \\ \hline\hline 2 & 3 & 3 & \end{array}$$

In considering possible transitions, remember that we may choose any pair of rows, any pair of columns, then act on the induced 2×2 table (keeping the induced small row/column sums of that 2×2 the same as they started). For a 3-rowed table there are 3 ways to choose a pair of rows, and for a 3-column table 3 ways to choose a pair of columns. So we have 3×3 different ways of choosing "2 rows and 2 columns".

Suppose our two rows were $\{1, 3\}$ and our two columns were $\{2, 3\}$. This picks out the following highlighted subtable,

$$\mathsf{X} = \begin{array}{ccc||c} \hline 2 & \mathbf{0} & \mathbf{0} & 2 \\ \hline 0 & 2 & 0 & 2 \\ \hline 0 & \mathbf{1} & \mathbf{3} & 4 \\ \hline\hline 2 & 3 & 3 & \end{array},$$

with the induced row sums $0, 4$ and the induced column sums $1, 3$. In this circumstance, with the empty first row, there is only one possible completion of the 2×2 subtable, which is the original assignment. So in the case where we choose rows $\{1, 3\}$ and columns $\{2, 3\}$, we can *only* transition to our original X.

Note that if we have chosen rows $1, 2$ and columns $2, 3$, we would again have the value 0 for row 1 of the induced subtable, and again can't move to any different state.

The same argument above can be applied to a situation where we choose columns $1, 3$, and we have row 2 as one of our rows. Then the induced sum on row 2 is 0, and again this "freezes" the original assignment, so again we are forced to stay at our initial X. Note this covers two different subtables, when we take columns $1, 3$, and we *either* choose rows $1, 2$ *or* we choose rows $2, 3$.

Examining column 1, we can see that any subtable with rows $2, 3$ and which has column 1 in its column set, will freeze the induced column sum for column 1m, and hence freeze the rest of the 2×2 subtable. This covers two cases of the subtable choice, rows $2, 3$ with *either* columns $1, 2$ *or* columns $1, 3$.

We have shown that 5 of the 9 2×2 subtable choices will freeze the table X (there was overlap between the scenarios that make column 1 induce 0 and make row 2 induce 0).

So there are 4 subtables left, that have some "freedom". These are rows $\{2, 3\}$, cols $\{2, 3\}$; rows $\{1, 2\}$, cols $\{1, 2\}$; rows $\{1, 3\}$, cols $\{1, 3\}$ and rows $\{1, 3\}$, cols $\{1, 2\}$.

In the first case of rows $\{2,3\}$, cols $\{2,3\}$, we get a little subtable with induced row sums $2,4$ and column sums $3,3$:

$$\mathsf{X} = \begin{array}{|ccc||c|} \hline 2 & 0 & 0 & 2 \\ \hline 0 & \mathbf{2} & \mathbf{0} & 2 \\ \hline 0 & \mathbf{1} & \mathbf{3} & 4 \\ \hline\hline 2 & 3 & 3 & \end{array}.$$

There are three values that could be placed in entry $(2,2)$ (any of 0, 1, 2) and each of those can be extended to a 2×2 subtable with row sums $2,4$ and colums sums $3,3$. So we have *two* new tables different from X that can be achieved with this choice of subtable, these are the following:

$$\mathsf{X} = \begin{array}{|ccc||c|} \hline 2 & 0 & 0 & 2 \\ \hline 0 & \mathbf{0} & \mathbf{2} & 2 \\ \hline 0 & \mathbf{3} & \mathbf{1} & 4 \\ \hline\hline 2 & 3 & 3 & \end{array},$$

$$\mathsf{X} = \begin{array}{|ccc||c|} \hline 2 & 0 & 0 & 2 \\ \hline 0 & \mathbf{1} & \mathbf{1} & 2 \\ \hline 0 & \mathbf{2} & \mathbf{2} & 4 \\ \hline\hline 2 & 3 & 3 & \end{array}.$$

We can continue on in this manner. If we had chosen rows $\{1,2\}$ and columns $\{1,2\}$, we would have a 2×2 subtable with induced row sums $2,2$ and induced column sums $2,2$, and in this situation there are again three completions of this subtable which preserve the row/column sums (giving position $(1,1)$ value 0, 1, or 2), two of which are different from X.

If we had chosen rows $\{1,3\}$, columns $\{1,3\}$, then we would have got a subtable with induced row sums $2,3$ and column sums also $2,3$, we again have three completions of this subtable which preserve the row/column sums (giving position $(1,1)$ value 0, 1, or 2), two of which are different from X.

Finally, had we chosen rows $\{1,3\}$, columns $\{1,2\}$ we would have got a subtable with induced row sums $2,1$ and column sums $2,1$ and in this case there are just two completions of this subtable which preserve the row/column sums (giving position $(3,2)$ value 0, 1), just one of which are different from X.

Note that the probability of $\mathsf{X} \to \mathsf{X}'$ for the first six of these different X' will be $\frac{1}{9}\frac{1}{3}$ (the probability of selecting that subtable, times the probability of that particular completion), and the 7th different X' (when we chose rows $\{1,3\}$, columns $\{1,2\}$) will have probability $\frac{1}{9}\frac{1}{2}$.

The probability that we have the transition $\mathsf{X} \to \mathsf{X}$ (ie, no change) is then $1 - 6 \cdot \frac{1}{9}\frac{1}{3} - \frac{1}{9}\frac{1}{2}$, which is $\frac{13}{18}$.

Mary Cryan, updated 18th April