Randomness and Computation

or, "Randomized Algorithms"

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Bounding deviation

We already have . . .

Theorem (3.1, Markov's Inequality)

Let X be any random variable that takes only non-negative values. Then for any $a > 0$,

$$\Pr[X \geq a] \;\leq\; \frac{\mathrm{E}[X]}{a}.$$

And also . . .

Theorem (3.2, Chebyshev's Inequality)

For every $a > 0$,

$$\Pr[|X - \mathrm{E}[X]| \geq a] \;\leq\; \frac{\mathrm{Var}[X]}{a^2}.$$

These are *generic*. Chernoff/Hoeffding bounds (specific) give tighter bounds for *sums of independent 0/1 variables* and related distributions.

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Chernoff Bounds from the book

Poisson trials - sequence of Bernoulli variables X_i with varying p_is.

Theorem (4.4)

Let $X_1, \dots, X_n$ be independent Poisson trials such that $\Pr[X_i = 1] = p_i$ for all $i \in [n]$. Let $X = \sum_{i=1}^{n} X_i$, and $\mu = \mathrm{E}[X]$. We have the following Chernoff bounds*:*

1. *For any $\delta > 0$,*

$$\Pr[X \geq (1+\delta)\mu] \; \leq \; \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{\mu};$$

2. *For any $0 < \delta \leq 1$,*

$$\Pr[X \geq (1+\delta)\mu] \; \leq \; e^{-\mu\delta^2/3};$$

3. *For $R \geq 6\mu$,*

$$\Pr[X \geq R] \; \leq 2^{-R}.$$

Chernoff Bounds from the book

Lemma

For n independent Poisson trials $X_1, \ldots, X_n$ and $X = \sum_{i=1}^{n} X_i$, $\mu = \mathrm{E}[X]$,

$$\mathrm{E}[e^{tX}] \ \le \ e^{\mu(e^t - 1)}.$$

Proof.

To prove the result, we will consider $\mathrm{E}[e^{tX}]$ for $t > 0$.

This is $\mathrm{E}[e^{t(\sum_{i=1}^{n} X_i)}] = E[\prod_{i=1}^{n} e^{tX_i}]$. The X_i and hence the e^{tX_i} are mutually independent, so by Thm 3.3, $\mathrm{E}[e^{tX}] = \prod_{i=1}^{n} \mathrm{E}[e^{tX_i}]$.

Each e^{tX_i} has expectation

$$\begin{aligned} \mathrm{E}[e^{tX_i}] &= p_i \cdot e^t + (1 - p_i) \cdot 1 \\ &= 1 + p_i(e^t - 1) \\ &\le e^{p_i(e^t - 1)} && \text{by } 1 + x \le e^x \text{ for } x \in \mathbb{R} \end{aligned}$$

$$\mathrm{E}[e^{tX}] \ \le \ \prod_{i=1}^{n} e^{p_i(e^t-1)} \ = \ e^{\sum_{i=1}^{n} p_i(e^t-1)} \ = \ e^{\mu(e^t-1)}.$$

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Proof of Thm 4.4 (1.)

Interested in events when $X \geq (1+\delta)\mu$.
Identical to when $e^X \geq e^{(1+\delta)\mu}$, or for any $t > 0$, when $e^{tX} \geq e^{t(1+\delta)\mu}$.

$$\begin{aligned}
\Pr[X \geq (1+\delta)\mu] &= \Pr[e^{tX} \geq e^{t(1+\delta)\mu}] \\
&\leq \frac{\mathrm{E}[e^{tX}]}{e^{t(1+\delta)\mu}} && \text{by Markov's Inequality} \\
&\leq \frac{e^{\mu(e^t-1)}}{e^{t(1+\delta)\mu}} && \text{by Lemma just proved}
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Now take $t = \ln(1+\delta)$ (and note this is > 0) to see

$$\begin{aligned}
\Pr[X \geq (1+\delta)\mu] &\leq \frac{e^{\mu(e^{\ln(1+\delta)}-1)}}{e^{\ln(1+\delta)(1+\delta)\mu}} \\
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Proof of Thm 4.4 (2.)

Already have

$$\Pr[X \geq (1+\delta)\mu] \quad \leq \quad \left(\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right)^{\mu}.$$

The rhs will be $\leq e^{-\mu\delta^2/3}$ if and only if (taking μ-th root, then $\ln$)

$$\delta - (1+\delta)\ln(1+\delta) < -\delta^2/3$$

We will show the following f is always negative for $\delta \in (0,1)$

$$f(\delta) \quad =_{def} \quad \delta - (1+\delta)\ln(1+\delta) + \delta^2/3$$

Differentiating,

$$f'(\delta) \quad = \quad 1 - \ln(1+\delta) - (1+\delta)\frac{1}{1+\delta} + \frac{2\delta}{3}$$

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Proof of Thm 4.4 (2.) cont'd.

$$f'(\delta) = -\ln(1+\delta) + \frac{2\delta}{3}.$$

Differentiating again

$$f''(\delta) = -\frac{1}{1+\delta} + \frac{2}{3} = -\frac{1}{1+\delta} + \frac{2}{3}$$

Note

$$f''(\delta) \begin{cases} < 0 & \text{for } 0 < \delta < 1/2 \\ 0 & \delta = 1/2 \\ > 0 & \delta > 1/2 \end{cases}$$

Also $f'(0) = 0$, $f'(1) < 0$ (check $\delta = 1$ in top equation), and by f' decreasing first, then increasing from $1/2$) $f'(\delta) < 0$ on $(0, 1)$.
By $f(0) = 0$, this implies that $f(\delta) \leq 0$ in all of $[0, 1]$.
Hence $\delta - (1+\delta)\ln(1+\delta) < -\delta^2/3$, proving 2.

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Chernoff Bounds from the book (other direction)

Theorem (4.5)

Let $X_1, \ldots, X_n$ be independent Poisson trials such that $\Pr[X_i = 1] = p_i$ for all $i \in [n]$. Let $X = \sum_{i=1}^{n} X_i$, and $\mu = \mathrm{E}[X]$. For any $0 < \delta < 1$, we have the following Chernoff bounds*:*

1. $$\Pr[X \leq (1-\delta)\mu] \;\leq\; \left(\frac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right)^{\mu};$$

2. $$\Pr[X \leq (1-\delta)\mu] \;\leq\; e^{-\mu\delta^2/2};$$

- Proof is similar to Thm 4.4.
- Bound of 2. is slightly *better* than for the $\geq (1+\delta)\mu$ bound.
- No 3. Why?

Concentration

Corollary (4.6)

Let $X_1, \ldots, X_n$ be independent Poisson trials such that $\Pr[X_i = 1] = p_i$ for all $i \in [n]$. Let $X = \sum_{i=1}^{n} X_i$, and $\mu = \mathrm{E}[X]$. Then for any $\delta, 0 < \delta < 1$,

$$\Pr[|X - \mu| \geq \delta\mu] \;\leq\; 2e^{-\mu\delta^2/3}.$$

- For almost all applications, we will want to work with a *symmetric* version like the Corollary.
- We “threw away" a bit in moving from the $\left(\frac{e^{\pm\delta}}{(1\pm\delta)^{1\pm\delta}}\right)^{\mu}$ versions, but they are tricky to work with.

Analysing a collection of coin flips

Suppose we have $p_i = 1/2$ for all $i \in [n]$.

We have $\mu = \mathrm{E}[X] = \frac{n}{2}$, $\mathrm{Var}[X] = \frac{n}{4}$.

Consider the probability of being further than $5\sqrt{n}$ from μ.

Chebyshev $\Pr[|X - \mu| \geq 5\sqrt{n}] \leq \frac{\mathrm{Var}[X]}{25n} = \frac{1}{100}$

Chernoff Work out the δ - we need $\mu\delta = 5\sqrt{n}$, so need $\delta = 5\sqrt{n}/\mu = 10\sqrt{n}/n = \frac{10}{\sqrt{n}}$. Then by Chernoff

$$\Pr[|X - \mu| \geq 5\sqrt{n}] \leq 2e^{-\mu\delta^2/3} = 2e^{\frac{-10^2 \cdot n}{2 \cdot 3 \cdot \sqrt{n}^2}} = 2e^{-16.6\ldots}.$$

This is much smaller than the Chebyshev bound (though note it doesn't depend on n).

Get much improved bounds because Chernoff uses specialised analysis for sums of independent Bernoulli variables.

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References

- Chapter 4 of “Probability and Computing"
- We will continue with Chernoff Bounds on Friday
- We may not have time to cover the packet routing analysis of 4.5. But it’s worth reading (but not examinable in the exam).