Randomness and Computation

or, "Randomized Algorithms"

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Bounding deviation

We already have ...

Theorem (3.1, Markov's Inequality)

Let X be any random variable that takes only non-negative values. Then for any $a > 0$,

$$\Pr[X \geq a] \;\leq\; \frac{\mathrm{E}[X]}{a}.$$

And also ...

Theorem (3.2, Chebyshev's Inequality)

For every $a > 0$,

$$\Pr[|X - \mathrm{E}[X]| \geq a] \;\leq\; \frac{\mathrm{Var}[X]}{a^2}.$$

These are *generic*. Chernoff/Hoeffding bounds (specific) give tighter bounds for *sums of independent 0/1 variables* and related distributions.

Chernoff Bounds from the book

Poisson trials - sequence of Bernoulli variables X_i with varying p_is.

Theorem (4.4)

Let $X_1, \ldots, X_n$ be independent Poisson trials such that $\Pr[X_i = 1] = p_i$ for all $i \in [n]$. Let $X = \sum_{i=1}^n X_i$, and $\mu = \mathrm{E}[X]$. We have the following Chernoff bounds*:*

1. *For any $\delta > 0$,*
$$\Pr[X \geq (1+\delta)\mu] \;\leq\; \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{\mu};$$
2. *For any $0 < \delta \leq 1$,*
$$\Pr[X \geq (1+\delta)\mu] \;\leq\; e^{-\mu\delta^2/3};$$
3. *For $R \geq 6\mu$,*
$$\Pr[X \geq R] \;\leq 2^{-R}.$$

Chernoff Bounds from the book

Lemma

For n independent Poisson trials $X_1, \ldots, X_n$ and $X = \sum_{i=1}^n X_i$, $\mu = \mathrm{E}[X]$,

$$\mathrm{E}[e^{tX}] \;\leq\; e^{\mu(e^t - 1)}.$$

Proof.

To prove the result, we will consider $\mathrm{E}[e^{tX}]$ for $t > 0$.

This is $\mathrm{E}[e^{t(\sum_{i=1}^n X_i)}] = E[\prod_{i=1}^n e^{tX_i}]$.The X_i and hence the e^{tX_i} are mutually independent, so by Thm 3.3, $\mathrm{E}[e^{tX}] = \prod_{i=1}^n \mathrm{E}[e^{tX_i}]$.

Each e^{tX_i} has expectation

$$\begin{aligned}
\mathrm{E}[e^{tX_i}] &= p_i \cdot e^t + (1 - p_i) \cdot 1 \\
&= 1 + p_i(e^t - 1) \\
&\leq e^{p_i(e^t - 1)} \qquad \text{by } 1 + x \leq e^x \text{ for } x \in \mathbb{R}
\end{aligned}$$

$$\mathrm{E}[e^{tX}] \leq \prod_{i=1}^n e^{p_i(e^t-1)} = e^{\sum_{i=1}^n p_i(e^t-1)} = e^{\mu(e^t-1)}.$$

Chernoff Bounds from the book

Proof of Thm 4.4 (1.)

Interested in events when $X \geq (1+\delta)\mu$.
Identical to when $e^X \geq e^{(1+\delta)\mu}$, or for any $t > 0$, when $e^{tX} \geq e^{t(1+\delta)\mu}$.

$$
\begin{aligned}
\Pr[X \geq (1+\delta)\mu] &= \Pr[e^{tX} \geq e^{t(1+\delta)\mu}] \\
&\leq \frac{\mathrm{E}[e^{tX}]}{e^{t(1+\delta)\mu}} && \text{by Markov's Inequality} \\
&\leq \frac{e^{\mu(e^t-1)}}{e^{t(1+\delta)\mu}} && \text{by Lemma just proved}
\end{aligned}
$$

Now take $t = \ln(1+\delta)$ (and note this is > 0) to see

$$
\begin{aligned}
\Pr[X \geq (1+\delta)\mu] &\leq \frac{e^{\mu(e^{\ln(1+\delta)}-1)}}{e^{\ln(1+\delta)(1+\delta)\mu}} \\
&= \frac{e^{\mu\delta}}{(1+\delta)^{(1+\delta)\mu}} = \left(\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right)^{\mu}.
\end{aligned}
$$

Chernoff Bounds from the book

Proof of Thm 4.4 (2.)

Already have

$$
\Pr[X \geq (1+\delta)\mu] \leq \left(\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right)^{\mu}.
$$

The rhs will be $\leq e^{-\mu\delta^2/3}$ if and only if (taking μ-th root, then ln)

$$
\delta - (1+\delta)\ln(1+\delta) < -\delta^2/3
$$

We will show the following f is always negative for $\delta \in (0,1)$

$$
f(\delta) =_{def} \delta - (1+\delta)\ln(1+\delta) + \delta^2/3
$$

Differentiating,

$$
\begin{aligned}
f'(\delta) &= 1 - \ln(1+\delta) - (1+\delta)\frac{1}{1+\delta} + \frac{2\delta}{3} \\
&= -\ln(1+\delta) + \frac{2\delta}{3}.
\end{aligned}
$$

Chernoff Bounds from the book

Proof of Thm 4.4 (2.) cont'd.

$$
f'(\delta) = -\ln(1+\delta) + \frac{2\delta}{3}.
$$

Differentiating again

$$
f''(\delta) = -\frac{1}{1+\delta} + \frac{2}{3} = -\frac{1}{1+\delta} + \frac{2}{3}
$$

Note

$$
f''(\delta) \begin{cases} < 0 & \text{for } 0 < \delta < 1/2 \\ 0 & \delta = 1/2 \\ > 0 & \delta > 1/2 \end{cases}
$$

Also $f'(0) = 0$, $f'(1) < 0$ (check $\delta = 1$ in top equation), and by f' decreasing first, then increasing from $1/2$) $f'(\delta) < 0$ on $(0,1)$.
By $f(0) = 0$, this implies that $f(\delta) \leq 0$ in all of $[0,1]$.
Hence $\delta - (1+\delta)\ln(1+\delta) < -\delta^2/3$, proving 2. □

Chernoff Bounds from the book (other direction)

Theorem (4.5)

Let $X_1, \ldots, X_n$ be independent Poisson trials such that $\Pr[X_i = 1] = p_i$ for all $i \in [n]$. Let $X = \sum_{i=1}^{n} X_i$, and $\mu = \mathrm{E}[X]$. For any $0 < \delta < 1$, we have the following Chernoff bounds*:*

1. $$\Pr[X \leq (1-\delta)\mu] \leq \left(\frac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right)^{\mu};$$
2. $$\Pr[X \leq (1-\delta)\mu] \leq e^{-\mu\delta^2/2};$$

- Proof is similar to Thm 4.4.
- Bound of 2. is slightly *better* than for the $\geq (1+\delta)\mu$ bound.
- No 3. Why?

Concentration

Corollary (4.6)

Let $X_1, \ldots, X_n$ be independent Poisson trials such that $\Pr[X_i = 1] = p_i$ for all $i \in [n]$. Let $X = \sum_{i=1}^{n} X_i$, and $\mu = \mathrm{E}[X]$. Then for any $\delta, 0 < \delta < 1$,

$$\Pr[|X - \mu| \geq \delta\mu] \;\leq\; 2e^{-\mu\delta^2/3}.$$

- For almost all applications, we will want to work with a *symmetric* version like the Corollary.
- We “threw away" a bit in moving from the $\left(\frac{e^{\pm\delta}}{(1\pm\delta)^{1\pm\delta}}\right)^{\mu}$ versions, but they are tricky to work with.

Analysing a collection of coin flips

Suppose we have $p_i = 1/2$ for all $i \in [n]$.
We have $\mu = \mathrm{E}[X] = \frac{n}{2}$, $\mathrm{Var}[X] = \frac{n}{4}$.
Consider the probability of being further than $5\sqrt{n}$ from μ.

Chebyshev $\Pr[|X - \mu| \geq 5\sqrt{n}] \leq \frac{\mathrm{Var}[X]}{25n} = \frac{1}{100}$

Chernoff Work out the δ - we need $\mu\delta = 5\sqrt{n}$, so need $\delta = 5\sqrt{n}/\mu = 10\sqrt{n}/n = \frac{10}{\sqrt{n}}$. Then by Chernoff

$$\Pr[|X - \mu| \geq 5\sqrt{n}] \leq 2e^{-\mu\delta^2/3} \;=\; 2e^{\frac{-10^2\cdot n}{2\cdot 3\cdot \sqrt{n}^2}} = 2e^{-16.6\ldots}.$$

This is much smaller than the Chebyshev bound (though note it doesn’t depend on n).

Get much improved bounds because Chernoff uses specialised analysis for sums of independent Bernoulli variables.

References

- Chapter 4 of “Probability and Computing"
- We will continue with Chernoff Bounds on Friday
- We may not have time to cover the packet routing analysis of 4.5. But it’s worth reading (but not examinable in the exam).