

Randomness and Computation

or, “Randomized Algorithms”

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The probabilistic method

The Probabilistic Method is a nonconstructive method of proof, primarily used in combinatorics and pioneered by Paul Erdős, for proving the existence of a desired kind of mathematical object. It works by showing that if we randomly choose objects from a specified class, the probability that the result has the desired property is greater than zero. This is enough to tell us that there must be at least one object with the desired property in the class.

Note that although this approach uses probability, the result (that some object with the property exists) will be definite, not “in probability”.

Slightly different theme to the rest of the results in this course, as we are concerned with showing *existence* (rather than constructing the object). However, sometimes we can derandomize/construct.

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Max Cuts in Graphs

Of interest is the Max Cut of a given graph (as well as “Min”):

Given an undirected, unweighted graph $G = (V, E)$ with $|V| = n$, $|E| = m$, compute a “max cut”; that is, a partition of E into two non-empty sets S , $V \setminus S$, such that the following quantity is maximized:

$$\{e = (u, v) : u \in S, v \in V \setminus S\}$$

Well-known as one of the classical NP-complete problems. So we believe there is no *polynomial-time* algorithm to compute this exactly (not in $\Theta(n^2 m)$, not in $\Theta(m^5 n^9)$ etc).

We will show that every graph $G = (V, E)$ has a cut of size *at least* $|E|/2$.

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Max Cuts in Graphs

Consider the following Algorithm:

Algorithm RANDOMCUT($G = (V, E)$)

1. $S \leftarrow \emptyset$
2. **for** every $v \in V$ in fixed order **do**
3. Draw a value b uniformly from $\{0, 1\}$.
4. **if** ($b = 1$) **then**
5. $S \leftarrow S \cup \{v\}$
6. **return** $S, V \setminus S$

We are going to analyse this algorithm and show that C_S (the number of edges between S and $V \setminus S$) has *expected size* at least $|E|/2$.

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Max Cuts in Graphs

Theorem (6.3)

For any given graph $G = (V, E)$, there is some cut $(S, V \setminus S)$ such that $|C_S| \geq |E|/2$.

Proof.

We show that the *expected* cardinality of C_S , $\mathbb{E}[|C_S|]$ is at least $|E|/2$ when S is a random subset of V . We can write

$$\mathbb{E}[|C_S|] = \frac{1}{2^n} \sum_{S: S \subseteq V} \sum_{e=(u,v) \in E} \mathbb{I}_{\{(u,v) \cap S = 1\}}.$$

Switching summations,

$$\mathbb{E}[|C_S|] = \sum_{e=(u,v) \in E} \frac{1}{2^n} \sum_{S: S \subseteq V} \mathbb{I}_{\{(u,v) \cap S = 1\}}.$$

For every $e \in E$, it has 4 options wrt a randomly generated S :
 $u, v \in S, u \in S, v \notin S, u \notin S, v \in S$, and $u \notin S, v \notin S$.
Probability 1/4 each.

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Probabilistic Method

- ▶ We did not analyse the probability that RANDOMCUT gives a good (high cardinality) cut, and are not going to do that yet.
- ▶ Can *de-randomise* the algorithm using conditional probabilities.
- ▶ The proof that every graph has a cut of cardinality $\geq |E|/2$ is a very very simple example of *the probabilistic method*.
- ▶ With the probabilistic method, we use randomness and the laws of expectation to prove that certain structures must exist.
- ▶ More later in the course.

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Max Cuts in Graphs

Proof cont.

Hence, for a fixed $e \in E$,

$$\frac{1}{2^n} \sum_{S: S \subseteq V} \mathbb{I}_{\{(u,v) \cap S = 1\}} = \frac{2}{4} = \frac{1}{2}.$$

Hence, summing over all $e \in E$,

$$\mathbb{E}[|C_S|] = \sum_{e=(u,v) \in E} \frac{1}{2} = \frac{|E|}{2},$$

as claimed.

Now switch back to thinking of this as the expectation over random S . If the *expected* size is $|E|/2$, then *certainly* there is at least one cut of at least that size.

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De-randomization

We derandomize via “conditional expectation”.

Our concern is the value of $|C_S|$, and the expected value of this quantity will change throughout the algorithm (as vertices get added to S or $V \setminus S$).

Our random algorithm considered the vertices in fixed order.

Let $x_1, x_2, \dots, x_k, \dots$ be the choices for the variables ($x_i = 1$ means that v_i is added to S , otherwise it's added to $V \setminus S$).

Our derandomization will construct a specific cut (again defined by x_1, \dots, x_k, \dots) of size $\geq \frac{|E|}{2}$ by making decisions for the vertices one-by-one. At each step we will ensure we choose x_{k+1} so that

$$\mathbb{E}[|C_S| \mid x_1, \dots, x_{k+1}] \geq \mathbb{E}[|C_S| \mid x_1, \dots, x_k].$$

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Derandomization cont'd.

Suppose we have considered v_1, \dots, v_k so far, and we have taken decisions x_1, \dots, x_k for these vertices.

Suppose (induction hypothesis) we know that $E[|C_S| \mid x_1, \dots, x_k] \geq E[|C_S|]$.

Let $A = \{v_i \mid x_i = 1, i \leq k\}$, $B = \{v_i \mid x_i = 0, i \leq k\}$.

Think about the (random) process for adding v_{k+1} (picture).

There are two choices for x_{k+1} , of equal probability.

Hence

$$E[|C_S| \mid x_1, \dots, x_k] = \frac{E[|C_S| \mid x_1, \dots, x_k, x_{k+1} = 1] + E[|C_S| \mid x_1, \dots, x_k, x_{k+1} = 0]}{2}.$$

So one of these expectations is at least as good as $E[|C_S| \mid x_1, \dots, x_k]$, which (by induction) is at least as good as $E[|C_S|] = \frac{|E|}{2}$.



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Derandomization cont'd.

How do we decide which of $E[|C_S| \mid x_1, \dots, x_k, x_{k+1} = 1]$, $E[|C_S| \mid x_1, \dots, x_k, x_{k+1} = 0]$ is larger?

We can easily compute $\{ (v_i, v_{k+1}) \in E, v_i \in B, i \leq k \}$ and $\{ (v_i, v_{k+1}) \in E, v_i \in A, i \leq k \}$ by examining the graph and the cut so far $((A, B))$.

If $\{ (v_i, v_{k+1}) \in E, v_i \in B, i \leq k \}$ is larger than $\{ (v_i, v_{k+1}) \in E, v_i \in A, i \leq k \}$, we set $x_{k+1} = 1$, else we set $x_{k+1} = 0$.

By Induction, we construct a cut guaranteed to be as large as $E[|C_S|] = \frac{|E|}{2}$.

Base case?



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Derandomization cont'd.

How do we decide which of $E[|C_S| \mid x_1, \dots, x_k, x_{k+1} = 1]$, $E[|C_S| \mid x_1, \dots, x_k, x_{k+1} = 0]$ is larger?

- ▶ If $x_{k+1} = 1$ (ie, v_{k+1} goes into A), then we add 1 to $|C_S|$ for every $(v_i, v_{k+1}) \in E$ with $v_i \in B$ (and $i \leq k$, obviously).
- ▶ If $x_{k+1} = 0$ (ie, v_{k+1} goes into B), then we add 1 to $|C_S|$ for every $(v_i, v_{k+1}) \in E$ with $v_i \in A$ (and $i \leq k$, obviously).
- ▶ For every v_i with $i > k + 1$, we add $\frac{1}{2}$ to $E[|C_S| \mid x_1, \dots, x_k]$ regardless of whether v_{k+1} gets added to A or B .

So the *difference* of conditional expectations satisfies

$$\begin{aligned} & E[|C_S| \mid x_1, \dots, x_k, x_{k+1} = 1] - E[|C_S| \mid x_1, \dots, x_k, x_{k+1} = 0] \\ &= |\{(v_i, v_{k+1}) \in E, v_i \in B, i \leq k\}| - |\{(v_i, v_{k+1}) \in E, v_i \in A, i \leq k\}| \end{aligned}$$



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References

Today's topic is from Sections 6.2, 6.3 of the book. We will return to the probabilistic method, and derandomization, in a couple of weeks.

TCS "cheat sheet" is always useful

<http://www.tug.org/texshowcase/cheat.pdf>

We will start work on Chernoff Bounds next week. It's a good idea to look at the early sections of Chapter 4.



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