Randomness and Computation

or, “Randomized Algorithms”

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new topic: The Probabilistic Method cont’d.

In Lecture 11 (continuing today) we saw the application of the Probabilistic Method:

- To allow us to set bounds on certain parameters that will ensure a randomly-drawn combinatorial object (from whatever pool of possible objects we are focusing on) has some desirable property with probability > 0.

  (our example property was that the edge 2-colouring of K_n graph would be without any monochromatic K_k subgraphs, assuming n is large enough wrt a lower bound)

  - The “probabilistic method” then allows us to infer that *at least one* of the combinatorial objects (from our pool) must have the desired property.
- Sometimes we can also *derandomize* this existence proof and actually *construct* an object satisfying the desired property.

  (need to have a de-composable (wrt deferred decisions) for drawing the random object, then apply conditional probabilities)

The Probabilistic Method cont’d.

Previous slide refers to “having some desirable property with probability > 0”.

In practice, often the approach will be to evaluate with an *expectation* rather than a *probability*.

Many of the examples of the probabilistic method we meet in RC involve showing that we can construct a combinatorial object (from some pool) that avoids having some banned sub-structure.

- We can consider the *expected number* of the banned substructures, when we draw an object from the sample pool;
- Sometimes it will be possible to evaluate the *expected number* of banned-substructures. If this is < 1, then the *probability* that there are some combinatorial objects that avoid all banned-substructures is > 0.

Second Moment Method

Examples of the probabilistic method so far only worked with *expectation*. If we had $\mathrm{E}[\cdot] < 1$ for the number of banned sub-structures, we knew there must be some object that has none of the banned sub-structures at all.

However, if we have $\mathrm{E}[\cdot] > 1$, things are less clear. There are definitely objects containing the banned sub-structures, but how *likely* they are is not clear.

Early on in the course we gave the definition of the *second moment* of a discrete random variable X, this being $\mathrm{E}[X^2]$. Then variance is $\mathrm{E}[X^2] - \mathrm{E}[X]^2$.

The second moment (with Chebyshev) can help us show that a typical sample is likely to have X close to $\mathrm{E}[X]$.

Second Moment Method

Theorem (Theorem 6.7)

For any integer-valued random variable X with positive expectation, we have

$$\Pr[X = 0] \;\le\; \frac{\mathrm{Var}[X]}{(\mathrm{E}[X])^2}.$$

Proof.

Really just a special case of Chebyshev's Inequality. We are interested in $\Pr[X = 0]$, which is equal to $\Pr[\mathrm{E}[X] - X = \mathrm{E}[X]]$. Also,

$$\Pr[\mathrm{E}[X] - X = \mathrm{E}[X]] \;\le\; \Pr[\mathrm{E}[X] - X \ge \mathrm{E}[X]] \;\le\; \Pr[|\mathrm{E}[X] - X| \ge \mathrm{E}[X]].$$

Then this final $\Pr[\cdot]$ fits the form for Chebyshev's Inequality with $a = \mathrm{E}[X]$, so applying Chebyshev gives us

$$\Pr[|\mathrm{E}[X] - X| \ge \mathrm{E}[X]] \;\le\; \frac{\mathrm{Var}[X]}{(\mathrm{E}[X])^2)},$$

and this right-hand side also bounds $\Pr[X = 0]$. □

Threshold for 4-cliques in $G_{n,p}$

We are interested in the random model $G_{n,p}$, where we draw a random graph on n vertices by independently doing a Bernoulli trial for each potential edge $(u, v), u \in V, v \in V \setminus \{u\}$, adding the edge with probability p, omitting that edge if the trial returns 0.

We are interested in whether the drawn graph $G \leftarrow G_{n,p}$ contains a 4-clique or not.

Clearly the graph is more likely to have a 4-clique if p has a higher value (and $G \leftarrow G_{n,p}$ is likely to have more edges).

We will show that there is a *threshold* for "$G \leftarrow G_{n,p}$ having a 4-clique" when p is either side of $\Theta(n^{-2/3})$.

Threshold for 4-cliques in $G_{n,p}$

Random model $G_{n,p}$, interested in whether $G \leftarrow G_{n,p}$ contains a 4-clique.

Theorem (6.8(a))

Suppose we have some probability sequence $p = p(n)$ such that $p(n) = o(n^{-2/3})$.
Then for any $\epsilon > 0$, for sufficiently large n, the graph $G \leftarrow G_{n,p}$ will contain a 4-clique with probability less than ϵ.

Proof. Recall that $p = p(n), p(n) = o(n^{-2/3})$ means that for every $\delta > 0$, there is some $n_\delta \in \mathbb{N}$ such that $p(n) < \delta \cdot n^{-2/3}$ for all $n \ge n_\delta$.

Let X denote the number of 4-cliques in $G \leftarrow G_{n,p}$.

Then $\mathrm{E}[X] = \mathrm{E}[\sum_{f \subseteq [n], |f|=4} X_f]$, where $X_f = 1$ if those 4 vertices form a clique, 0 otherwise.

Linearity of exp. gives $\mathrm{E}[X] = \sum_{f \subseteq [n], |f|=4} \mathrm{E}[X_f]$.

Threshold for 4-cliques in $G_{n,p}$

Proof of 6.8 (a) cont'd.

Now we compute $\mathrm{E}[X_f]$ for a specific subset $f = \{u, v, w, x\}$.

These 4 vertices form a clique $\Leftrightarrow$ all 6 edges are in $G \leftarrow G_{n,p}$. This happens with probability p^6.

This value of $\mathrm{E}[X_f]$ is independent of the particular f, and there are exactly $\binom{n}{4}$ subsets satisfying $f \subseteq [n], |f| = 4$. Hence

$$\mathrm{E}[X] = \binom{n}{4} p^6 = \tfrac{n(n-1)(n-2)(n-3)}{24} \cdot p^6.$$

Now consider $\delta = (24\epsilon)^{1/6}$ in the definition of $o(n^{-2/3})$; then for $n \ge n_\delta$ we have $p \le \delta \cdot n^{-2/3}$, then $p^6 \le 24\epsilon (n^{-2/3})^6 = 24\epsilon n^{-4}$. Then $\mathrm{E}[X] \le \frac{n(n-1)(n-2)(n-3)}{24} \cdot p^6 < \epsilon$.

Certainly $\Pr[X \ge 1] \le \mathrm{E}[X] \le \epsilon$, as claimed.

Threshold for 4-cliques in $G_{n,p}$

Theorem (6.8(b))

Suppose we have some probability sequence $p = p(n)$ such that $p(n) = \omega(n^{-2/3})$.
Then for any $\epsilon > 0$, for sufficiently large n the graph $G \leftarrow G_{n,p}$ will contain a 4-clique with probability greater than $1 - \epsilon$.

Proof. Recall that $p = p(n), p(n) = \omega(n^{-2/3})$ means that for every $\delta > 0$, there is some $n_\delta \in \mathbb{N}$ such that $p = p(n) > \delta \cdot n^{-2/3}$ for all $n \geq n_\delta$.

Let X denote the number of 4-cliques in $G \leftarrow G_{n,p}$.

We know that $\mathrm{E}[X] = \binom{n}{4}p^6$ and by a similar argument to before, if $n > n_\delta$ of $\omega(n^{-2/3})$ then $\mathrm{E}[X] = \binom{n}{4} \cdot p^6 = \omega(1) \to \infty$ as $n \to \infty$.

This means $\mathrm{E}[X]$; however, it doesn't imply a lower bound for $\Pr[X > 0]$; need to examine the second moment.

Threshold for 4-cliques in $G_{n,p}$

Proof of 6.8(b) cont'd.

We want to calculate $\mathrm{Var}[X]$, however, not all the X_f variables are independent, so can't apply $\mathrm{Var}[X] = \sum_{f \subseteq [n], |f|=4} \mathrm{Var}[X_f]$.

We can apply the result from our early lectures to rewrite:

$$\mathrm{Var}[X] \;\leq\; \mathrm{E}[X] + \sum_{f \subseteq [n], |f|=4} \left[\sum_{g \subseteq [n], g \neq f, |g|=4} \mathrm{Cov}[X_f X_g] \right].$$

case (a): $|f \cap g| \leq 1$:
In this case f and g share no edges at all; and $\mathrm{E}[X_f X_g]$ is $\mathrm{E}[X_f]\mathrm{E}[X_g]$, hence $\mathrm{Cov}[X_f X_g] = 0$. Most likely case, there are $\binom{n}{4}\binom{n-4}{4} + n \cdot \binom{n-1}{3}\binom{n-4}{3}$ pairs like this. But their contribution to the "double sum" is 0, we can ignore.

case (b): $|f \cap g| = 2$:
Then f and g share one edge, and $\mathrm{E}[X_f X_g] = p^{11}$, and hence $\mathrm{Cov}[X_f X_g] = p^{11} - (p^6)^2 \leq p^{11}$. There are $\binom{n}{2}\binom{n-2}{2}\binom{n-4}{2}$ pairs like this.

Threshold for 4-cliques in $G_{n,p}$

Proof of 6.8(b) cont'd.

case (c): $|f \cap g| = 3$:
Then f and g share three edges, and $\mathrm{E}[X_f X_g] = p^3 \cdot (p^3)^2 = p^9$.
Hence $\mathrm{Cov}[X_f X_g] \leq p^9$. There are $\binom{n}{3}(n-3)(n-4)$ pairs like this.

Putting it all together ...

$$\begin{aligned} \sum_{f \subseteq [n], |f|=4} \; \sum_{g \subseteq [n], g \neq f, |g|=4} \mathrm{E}[X_f X_g] &\leq \binom{n}{3}\binom{n-3}{1}\binom{n-4}{1}p^9 + \binom{n}{2}\binom{n-2}{2}\binom{n-4}{2}p^{11} \\ &= O(n^4) \cdot p^6 + O(n^5) \cdot p^9 + O(n^6) \cdot p^{11} \end{aligned}$$

Note that given $p = \omega(n^{-2/3})$, both of these terms is "little-o" of $(\mathrm{E}[X])^2 = \Theta(p^{12} \cdot n^8)$.
Also adding $\mathrm{E}[X] = p^6\binom{n}{4}$ to the "double sum", this is also "little-o" of $\Theta(p^{12} \cdot n^8)$; hence $\mathrm{Var}[X]$ is $o(p^{12} \cdot n^8)$, and applying Chebyshev we find that

$$\Pr[X = 0] \leq \frac{o(p^{12} \cdot n^8)}{\Theta(p^{12} \cdot n^8)},$$

which tends to 0 as $n \to \infty$.

Notes

Reading

- You will want to read Sections 6.4, 6.5, 6.6, 6.7 from the book.

Doing

- Tutorial sheet for 5th, 6th March (week 7).
- You will be getting the coursework 2 specification next week.