# Randomness and Computation

or, “Randomized Algorithms”

Mary Cryan

School of Informatics
University of Edinburgh



## Balls in Bins

- $m$ balls, $n$ bins, and balls thrown *uniformly at random* into bins (usually one at a time).
- Magic bins with no upper limit on capacity.
- Common model of random allocations and their affect on overall *load* and *load balance*, typical *distribution* in the system.
- “Classic" question - what does the distribution look like for $m = n$? Max load? (*with high probability* results are what we want).
- We have already shown that when $m = n$ (same number of balls as bins) and $n$ if sufficiently large, the maximum load is $\leq \frac{3\ln(n)}{\ln\ln(n)}$ with probability at least $1 - \frac{1}{n}$.

  We will show an $\Omega(\frac{\ln(n)}{\ln\ln(n)})$ bound today.



## Some preliminary observations, definitions

The probability is of a specific bin (bin $i$, say) being empty:

$$(1 - \frac{1}{n})^m \sim e^{-m/n}.$$

Expected number of empty bins: $\sim ne^{-m/n}$
Probability $p_r$ of a specific bin having $r$ balls:

$$p_r = \binom{m}{r}\frac{1}{n}^r\left(1 - \frac{1}{n}\right)^{m-r}.$$

Note

$$p_r \sim \frac{e^{-m/n}}{r!}\frac{m^r}{n}.$$

### Definition (5.1)

A discrete *Poisson random variable* $X$ with parameter $\mu$ is given by the following probability distribution on $j = 0, 1, 2, \ldots$:

$$\Pr[X = j] = \frac{e^{-\mu}\mu^j}{j!}.$$



## Poisson as the limit of the Binomial Distribution

### Theorem (5.5)

*If $X_n$ is a binomial random variable with parameters $n$ and $p = p(n)$ such that $\lim_{n\to\infty} np = \lambda$ is a constant (independent of $n$), then for any fixed $k \in \mathbb{N}_0$*

$$\lim_{n\to\infty} \Pr[X_n = k] = \frac{e^{-\lambda}\lambda^k}{k!}.$$

## Poisson modelling of balls-in-bins

Our balls in bins model has $n$ bins, $m$ (for variable $m$) balls, and the balls are thrown into bins independently and uniformly at random.

Each bin $X_i^{(m)}$ behaves like a binomial r.v $B(m, \frac{1}{n})$.

Write $(X_1^{(m)}, \dots, X_n^{(m)})$ for the joint distribution (note the various $X_i^{(m)}$s are *not* independent).

For the "Poisson approximation" we take $\lambda = \frac{m}{n}$, and write $Y_i^{(m)}$ to denote a Poisson r.v with parameter $\lambda = m/n$.

We write $(Y_1^{(m)}, \dots, Y_n^{(m)})$ to denote a joint distribution of *independent* Poisson r.vs which are all independent.



## Some preliminaries

### Theorem (5.7)

*Let $f(x_1, \dots, x_n)$ be a non-negative function. Then*

$$\mathrm{E}[f(X_1^{(m)}, \dots, X_n^{(m)})] \leq e\sqrt{m} \cdot \mathrm{E}[f(Y_1^{(m)}, \dots, Y_n^{(m)})].$$

### Corollary (5.9)

*Any event that takes place with probability $p$ in the "Poisson case" takes place with probability at most $pe\sqrt{m}$ in the exact balls-in-bins case.*



## Lower bound for $n$ "balls in bins"

### Lemma (5.12)

*Let $n$ balls be thrown independently and uniformly at random into $n$ bins. Then (for $n$ sufficiently large) the maximum load is at least $\ln(n)/\ln\ln(n)$ with probability at least $1 - \frac{1}{n}$.*

### Proof.

For the Poisson variables, we have $\lambda = \frac{n}{n} = 1$. Let $M = \lceil \frac{\ln(n)}{\ln\ln(n)} \rceil$. For any bin $i$ (say),

$$\begin{aligned} & \Pr_{Poiss}[\text{bin } i \text{ has load } \geq M] \\ \geq\ & \Pr_{Poiss}[\text{bin } i \text{ has load } = M] \\ =\ & \frac{1^M e^{-1}}{M!} = \frac{1}{eM!} \end{aligned}$$

In our Poisson model, the bins are independent, so the probability *no bin has load* $\geq M$ (our bad event) is at most

$$\left(1 - \frac{1}{eM!}\right)^n \leq e^{-n/(eM!)}.$$



## Lower bound for $n$ "balls in bins"

### Proof of Lemma 5.1 cont'd.

We now relate $\Pr_{Poiss}[\text{bin } i \text{ has load } \geq M]$ to the probability of the same event in the balls-in-bins model.

Corollary 5.9 tells us that when we consider the exact balls-in-bins distribution $(X_1^{(n)}, \dots, X_n^{(n)})$, that the probability of the event "no bin has $\geq M$ balls" is at most

$$e\sqrt{n} \cdot e^{-n/(eM!)}.$$

We want this less than $n^{-1}$, ie we want $e^{1-n/(eM!)} \leq n^{-3/2}$. Taking $\ln(\cdot)$ of both sides, this happens if

$$\left(1 - \frac{n}{eM!}\right) \leq -\frac{3}{2}\ln(n) \Leftrightarrow 1 + \frac{3}{2}\ln(n) \leq \frac{n}{eM!}.$$

Now $M! \leq e\sqrt{M}(\frac{M}{e})^M \leq M(\frac{M}{e})^M$ (Lemma 5.8), hence $\frac{n}{eM!} \geq \frac{ne^M}{eM^{M+1}}$. □

## Lower bound for $n$ “balls in bins”

### Proof of Lemma 5.1 cont’d.

Therefore it will suffice to show that $1 + \frac{3}{2}\ln(n) \leq \frac{ne^M}{eM^{M+1}}$, or (for sufficiently large $n$), that

$$2\ln(n) \leq \frac{ne^M}{eM^{M+1}}.$$

Taking the $\ln$ of both sides, this happens (using $M \sim \frac{\ln(n)}{\ln\ln(n)}$) when

$$\ln(2)+\ln\ln(n) \leq \left(\ln(n) + \frac{\ln(n)}{\ln\ln(n)}\right) - \left(1 + \left(\frac{\ln(n)}{\ln\ln(n)} + 1\right)(\ln\ln(n) - \ln\ln\ln(n))\right),$$

ie, exactly when

$$1+\ln(2)+\ln\ln(n) \leq \left(\ln(n) + \frac{\ln(n)}{\ln\ln(n)}\right) - \ln(n) - \ln\ln(n) + \frac{\ln(n)\ln\ln\ln(n)}{\ln\ln(n)} + \ln\ln\ln(n),$$

ie, exactly when

$$1 + \ln(2) + 2\ln\ln(n) \leq \frac{\ln(n)}{\ln\ln(n)} + \frac{\ln(n)\ln\ln\ln(n)}{\ln\ln(n)} + \ln\ln\ln(n).$$



## Lower bound for $n$ “balls in bins”

### Proof of Lemma 5.1 cont’d.

To show that

$$1 + \ln(2) + 2\ln\ln(n) \leq \frac{\ln(n)}{\ln\ln(n)} + \frac{\ln(n)\ln\ln\ln(n)}{\ln\ln(n)} + \ln\ln\ln(n)$$

we will multiply across by $\ln\ln(n)$, to verify the equivalent inequality

$$(1+\ln(2))\ln\ln(n) + 2(\ln\ln(n))^2 \leq \ln(n) + \ln(n)\ln\ln\ln(n) + \ln\ln\ln(n)\ln\ln(n).$$

At this point we notice that we have two terms on the right ($\ln(n)$ and $\ln(n)\ln\ln\ln(n)$) which are exponentially larger than the two terms on the lhs - both lhs terms only grow wrt $\ln\ln(n)$.

We do not need to check the numbers - as $n$ grows the rhs will certainly be greater than the lhs.

Hence our claim holds. □



## References and Exercises

- Sections 5.1, 5.2 of “Probability and Computing".
- Sections 5.3 and 5.4 have all precise details of our $\Omega(\frac{\ln(n)}{\ln\ln(n)})$ result.
- Section 5.5 on Hashing is worth a read and has none of the Poisson stuff (I’m skipping it because of time limitations).

### Exercises

I will release a tutorial sheet.