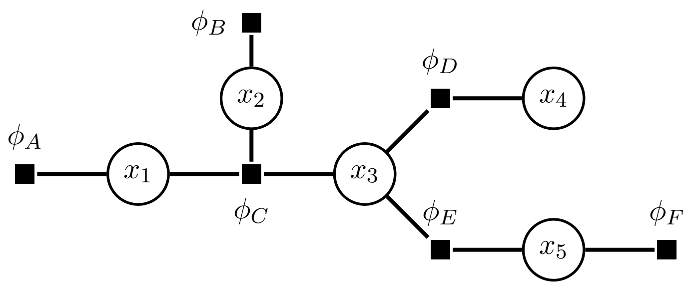
Exercises for the tutorials: 1(a-c) and 3(a-b).

The other exercises are for self-study and exam preparation. All material is examinable unless otherwise mentioned.

Exercise 1. *Sum-product message passing*

We here re-consider the factor tree from the lecture on exact inference.



Let all variables be binary, $x_i \in \{0, 1\}$, and the factors be defined as follows:

x_1	ϕ_A
0	2
1	4

x_2	ϕ_B
0	4
1	4

x_1	x_2	x_3	ϕ_C
0	0	0	4
1	0	0	2
0	1	0	2
1	1	0	6
0	0	1	2
1	0	1	6
0	1	1	6
1	1	1	4

x_3	x_4	ϕ_D
0	0	8
1	0	2
0	1	2
1	1	6

x_3	x_5	ϕ_E
0	0	3
1	0	6
0	1	6
1	1	3

x_5	ϕ_F
0	1
1	8

(a) Mark the graph with arrows indicating all messages that need to be computed for the computation of $p(x_1)$.

(b) Compute the messages that you have identified.

Assuming that the computation of the messages is scheduled according to a common clock, group the messages together so that all messages in the same group can be computed in parallel during a clock cycle.

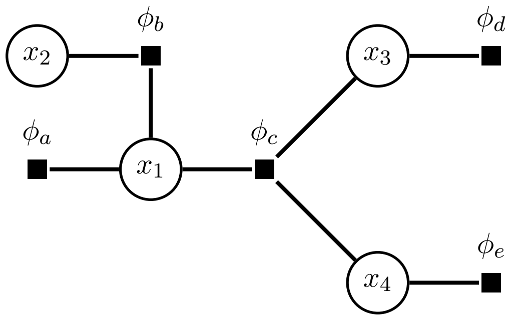
(c) What is $p(x_1 = 1)$?

(d) Draw the factor graph corresponding to $p(x_1, x_3, x_4, x_5 | x_2 = 1)$ and provide the numerical values for all factors.

(e) Compute $p(x_1 = 1 | x_2 = 1)$, re-using messages that you have already computed for the evaluation of $p(x_1 = 1)$.

Exercise 2. ***Sum-product message passing***

The following factor graph represents a Gibbs distribution over four binary variables $x_i \in \{0, 1\}$.



The factors ϕ_a, ϕ_b, ϕ_d are defined as follows:

x_1	ϕ_a
0	2
1	1

x_1	x_2	ϕ_b
0	0	5
1	0	2
0	1	2
1	1	6

x_3	ϕ_d
0	1
1	2

and $\phi_c(x_1, x_3, x_4) = 1$ if $x_1 = x_3 = x_4$, and is zero otherwise.

For all questions below, justify your answer:

(a) Compute the values of $\mu_{x_2 \to \phi_b}(x_2)$ for $x_2 = 0$ and $x_2 = 1$.

(b) Assume the message $\mu_{x_4 \to \phi_c}(x_4)$ equals

$$\mu_{x_4 \to \phi_c}(x_4) = \begin{cases} 1 & \text{if } x_4 = 0 \\ 3 & \text{if } x_4 = 1 \end{cases}$$

Compute the values of $\phi_e(x_4)$ for $x_4 = 0$ and $x_4 = 1$.

(c) Compute the values of $\mu_{\phi_c \to x_1}(x_1)$ for $x_1 = 0$ and $x_1 = 1$.

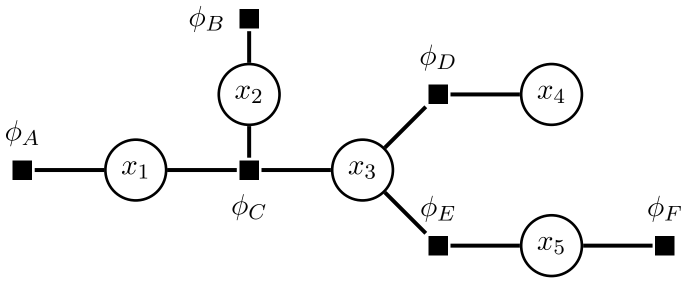
(d) The message $\mu_{\phi_b \to x_1}(x_1)$ equals

$$\mu_{\phi_b \to x_1}(x_1) = \begin{cases} 7 & \text{if } x_1 = 0 \\ 8 & \text{if } x_1 = 1 \end{cases}$$

What is the probability that $x_1 = 1$, i.e. $p(x_1 = 1)$?

Exercise 3. ***Max-sum message passing***

We here compute most probable states for the factor graph and factors below.



Let all variables be binary, $x_i \in \{0, 1\}$, and the factors be defined as follows:

x_1	ϕ_A
0	2
1	4

x_2	ϕ_B
0	4
1	4

x_1	x_2	x_3	ϕ_C
0	0	0	4
1	0	0	2
0	1	0	2
1	1	0	6
0	0	1	2
1	0	1	6
0	1	1	6
1	1	1	4

x_3	x_4	ϕ_D
0	0	8
1	0	2
0	1	2
1	1	6

x_3	x_5	ϕ_E
0	0	3
1	0	6
0	1	6
1	1	3

x_5	ϕ_F
0	1
1	8

(a) Will we need to compute the normalising constant Z to determine $\text{argmax}_{\mathbf{x}}\, p(x_1, \ldots, x_5)$?

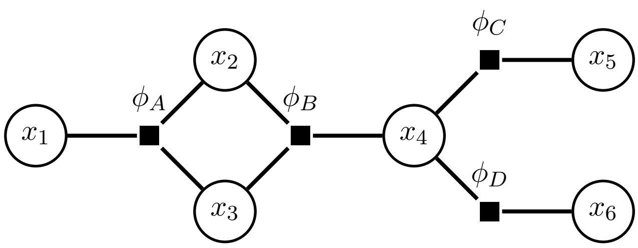
(b) Compute $\text{argmax}_{x_1,x_2,x_3}\, p(x_1, x_2, x_3 | x_4 = 0, x_5 = 0)$ via max-sum message passing.

(c) Compute $\text{argmax}_{x_1,\ldots,x_5}\, p(x_1, \ldots, x_5)$ via max-sum message passing with x_1 as root.

(d) Compute $\text{argmax}_{x_1,\ldots,x_5}\, p(x_1, \ldots, x_5)$ via max-sum message passing with x_3 as root.

Exercise 4. ***Choice of elimination order in factor graphs***

Consider the following factor graph, which contains a loop:



Let all variables be binary, $x_i \in \{0, 1\}$, and the factors be defined as follows:

x_1	x_2	x_3	ϕ_A
0	0	0	4
1	0	0	2
0	1	0	2
1	1	0	6
0	0	1	2
1	0	1	6
0	1	1	6
1	1	1	4

x_2	x_3	x_4	ϕ_B
0	0	0	2
1	0	0	2
0	1	0	4
1	1	0	2
0	0	1	6
1	0	1	8
0	1	1	4
1	1	1	2

x_4	x_5	ϕ_C
0	0	8
1	0	2
0	1	2
1	1	6

x_4	x_6	ϕ_D
0	0	3
1	0	6
0	1	6
1	1	3

(a) Draw the factor graph corresponding to $p(x_2, x_3, x_4, x_5 \mid x_1 = 0, x_6 = 1)$ and give the tables defining the new factors $\phi_A^{x_1=0}(x_2, x_3)$ and $\phi_D^{x_6=1}(x_4)$ that you obtain.

(b) Find $p(x_2 \mid x_1 = 0, x_6 = 1)$ using the elimination ordering (x_4, x_5, x_3):

(i) Draw the graph for $p(x_2, x_3, x_5 \mid x_1 = 0, x_6 = 1)$ by marginalising x_4
Compute the table for the new factor $\tilde{\phi}_4(x_2, x_3, x_5)$

(ii) Draw the graph for $p(x_2, x_3 \mid x_1 = 0, x_6 = 1)$ by marginalising x_5
Compute the table for the new factor $\tilde{\phi}_{45}(x_2, x_3)$

(iii) Draw the graph for $p(x_2 \mid x_1 = 0, x_6 = 1)$ by marginalising x_3
Compute the table for the new factor $\tilde{\phi}_{453}(x_2)$

(c) Now determine $p(x_2 \mid x_1 = 0, x_6 = 1)$ with the elimination ordering (x_5, x_4, x_3):

(i) Draw the graph for $p(x_2, x_3, x_4, \mid x_1 = 0, x_6 = 1)$ by marginalising x_5
Compute the table for the new factor $\tilde{\phi}_5(x_4)$

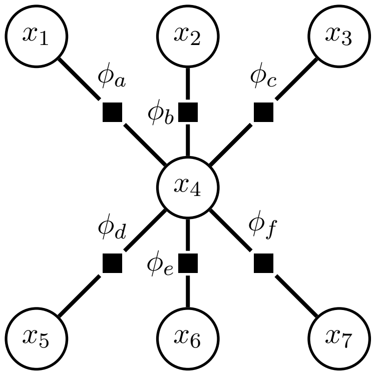
(ii) Draw the graph for $p(x_2, x_3 \mid x_1 = 0, x_6 = 1)$ by marginalising x_4
Compute the table for the new factor $\tilde{\phi}_{54}(x_2, x_3)$

(iii) Draw the graph for $p(x_2 \mid x_1 = 0, x_6 = 1)$ by marginalising x_3
Compute the table for the new factor $\tilde{\phi}_{543}(x_2)$

(d) Which variable ordering, (x_4, x_5, x_3) or (x_5, x_4, x_3) do you prefer?

Exercise 5. ***Choice of elimination order in factor graphs***

We would like to compute the marginal $p(x_1)$ by variable elimination for a joint pmf represented by the following factor graph. All variables x_i can take K different values.



(a) A friend proposes the elimination order $x_4, x_5, x_6, x_7, x_3, x_2$, i.e. to do x_4 first and x_2 last. Explain why this is computationally inefficient.

(b) Propose an elimination ordering that achieves $O(K^2)$ computational cost per variable elimination and explain why it does so.