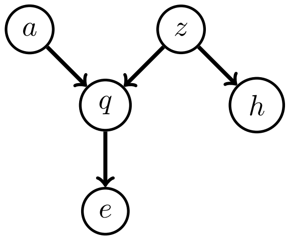
Exercises for the tutorials: 1, 2(a-b), 3.

The other exercises are for self-study and exam preparation. All material is examinable unless otherwise mentioned.

Exercise 1. *Directed graph concepts*

We here consider the directed graph below that was partly discussed in the lecture.



(a) List all trails in the graph (of maximal length)

(b) List all directed paths in the graph (of maximal length)

(c) What are the descendants of z?

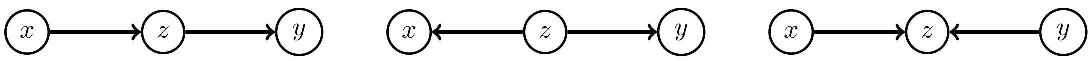
(d) What are the non-descendants of q?

(e) Which of the following orderings are topological to the graph?

- (a,z,h,q,e)
- (a,z,e,h,q)
- (z,a,q,h,e)
- (z,q,e,a,h)

Exercise 2. *Canonical connections*

We here derive the independencies that hold in the three canonical connections that exist in DAGs, shown in Figure 1.



(a) Serial connection (b) Diverging connection (c) Converging connection

Figure 1: The three canonical connections in DAGs.

(a) For the serial connection, use the ordered Markov property to show that $x \perp\!\!\!\perp y \mid z$.

(b) For the serial connection, show that the marginal $p(x, y)$ does generally not factorise into $p(x)p(y)$, i.e. that $x \perp\!\!\!\perp y$ does not hold.

(c) For the diverging connection, use the ordered Markov property to show that $x \perp\!\!\!\perp y \mid z$.

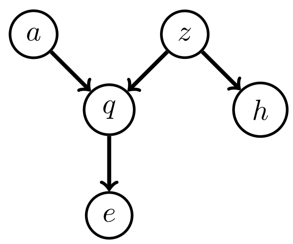
(d) For the diverging connection, show that the marginal $p(x, y)$ does generaly not factorise into $p(x)p(y)$, i.e. that $x \perp\!\!\!\perp y$ does not hold.

(e) For the converging connection, show that $x \perp\!\!\!\perp y$.

(f) For the converging connection, show that $x \perp\!\!\!\perp y \mid z$ does generally not hold.

Exercise 3. ***Ordered and local Markov properties, d-separation***

We continue with the investigation of the graph from Exercise 1 shown below for reference.



(a) The ordering (z, h, a, q, e) is topological to the graph. What are the independencies that follow from the ordered Markov property?

(b) What are the independencies that follow from the local Markov property?

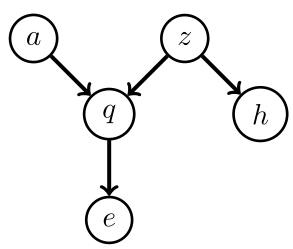
(c) The independency relations obtained via the ordered and local Markov property include $q \perp\!\!\!\perp h \mid \{a, z\}$. Verify the independency using d-separation.

(d) Use d-separation to check whether $a \perp\!\!\!\perp h \mid e$ holds.

(e) Assume all variables in the graph are binary. How many numbers do you need to specify, or learn from data, in order to fully specify the probability distribution?

Exercise 4. ***More on ordered and local Markov properties, d-separation***

We continue with the investigation of the graph below



(a) Why can the ordered or local Markov property not be used to check whether $a \perp\!\!\!\perp h \mid e$ may hold?

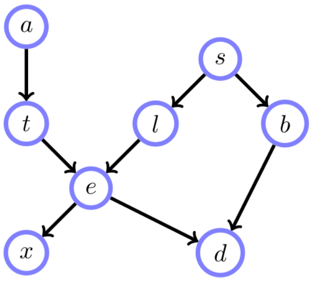
(b) The independency relations obtained via the ordered and local Markov property include $a \perp\!\!\!\perp \{z, h\}$. Verify the independency using d-separation.

(c) Determine the Markov blanket of z.

(d) Verify that $q \perp\!\!\!\perp h \mid \{a, z\}$ holds by manipulating the probability distribution induced by the graph.

Exercise 5. ***Chest clinic (based on Barber's exercise 3.3)***

The directed graphical model in Figure 2 is about the diagnosis of lung disease (t=tuberculosis or l=lung cancer). In this model, a visit to some place "a" is thought to increase the probability of tuberculosis.



x	Positive X-ray
d	Dyspnea (shortness of breath)
e	Either tuberculosis or lung cancer
t	Tuberculosis
l	Lunc cancer
b	Bronchitis
a	Visited place a
s	Smoker

Figure 2: Graphical model for Exercise 5 (Barber Figure 3.15).

(a) Explain which of the following independence relationships hold for all distributions that factorise over the graph.

1. $t \perp\!\!\!\perp s \mid d$
2. $l \perp\!\!\!\perp b \mid s$

(b) Can we simplify $p(l|b, s)$ to $p(l|s)$?

Exercise 6. ***More on the chest clinic (based on Barber's exercise 3.3)***

Consider the directed graphical model in Figure 2.

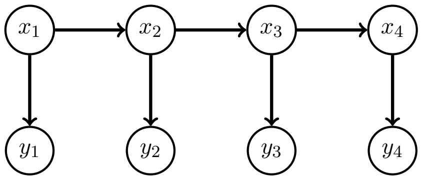
(a) Explain which of the following independence relationships hold for all distributions that factorise over the graph.

1. $a \perp\!\!\!\perp s \mid l$
2. $a \perp\!\!\!\perp s \mid l, d$

(b) Let g be a (deterministic) function of x and t. Is the expected value $\mathbb{E}[g(x, t) \mid l, b]$ equal to $\mathbb{E}[g(x, t) \mid l]$?

Exercise 7. ***Hidden Markov models***

This exercise is about directed graphical models that are specified by the following DAG:



These models are called "hidden" Markov models because we typically assume to only observe the y_i and not the x_i that follow a Markov model.

(a) Show that all probabilistic models specified by the DAG factorise as

$$p(x_1, y_1, x_2, y_2, \ldots, x_4, y_4) = p(x_1)p(y_1|x_1)p(x_2|x_1)p(y_2|x_2)p(x_3|x_2)p(y_3|x_3)p(x_4|x_3)p(y_4|x_4)$$

(b) Derive the independencies implied by the ordered Markov property with the topological ordering $(x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4)$

(c) Derive the independencies implied by the ordered Markov property with the topological ordering $(x_1, x_2, \ldots, x_4, y_1, \ldots, y_4)$.

(d) Does $y_4 \perp\!\!\!\perp y_1 \mid y_3$ hold?

Exercise 8. ***Alternative characterisation of independencies***

We have seen that $x \perp\!\!\!\perp y|z$ is characterised by $p(x, y|z) = p(x|z)p(y|z)$ or, equivalently, by $p(x|y, z) = p(x|z)$. Show that further equivalent characterisations are

$$p(x, y, z) = p(x|z)p(y|z)p(z) \quad \text{and} \tag{1}$$

$$p(x, y, z) = a(x, z)b(y, z) \quad \text{for some non-neg. functions } a(x, z) \text{ and } b(x, z). \tag{2}$$

The characterisation in Equation (2) will be important for undirected graphical models.

Exercise 9. ***More on independencies***

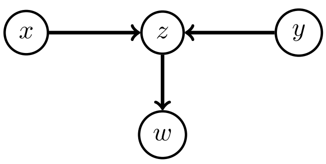
This exercise is on further properties and characterisations of statistical independence.

(a) Without using d-separation, show that $x \perp\!\!\!\perp \{y, w\} \mid z$ implies that $x \perp\!\!\!\perp y \mid z$ and $x \perp\!\!\!\perp w \mid z$. *Hint: use the definition of statistical independence in terms of the factorisation of pmfs/pdfs.*

(b) For the directed graphical model below, show that the following two statements hold without using d-separation:

$$x \perp\!\!\!\perp y \quad \text{and} \tag{3}$$

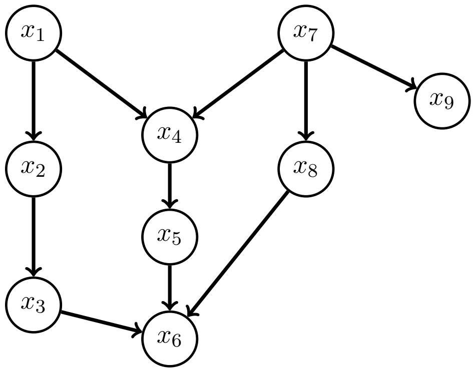
$$x \not\perp\!\!\!\perp y \mid w \tag{4}$$



The exercise shows that not only conditioning on a collider node but also on one of its descendents activates the trail between x and y. You can use the result that $x \perp\!\!\!\perp y|w \Leftrightarrow p(x, y, w) = a(x, w)b(y, w)$ for some non-negative functions $a(x, w)$ and $b(y, w)$.

Exercise 10. ***Independencies in directed graphical models***

Consider the following directed acyclic graph.



For each of the statements below, determine whether it holds for all probabilistic models that factorise over the graph. Provide a justification for your answer.

(a) $p(x_7|x_2) = p(x_7)$

(b) $x_1 \not\perp\!\!\!\perp x_3$

(c) $p(x_1, x_2, x_4) \propto \phi_1(x_1, x_2)\phi_2(x_1, x_4)$ for some non-negative functions ϕ_1 and ϕ_2.

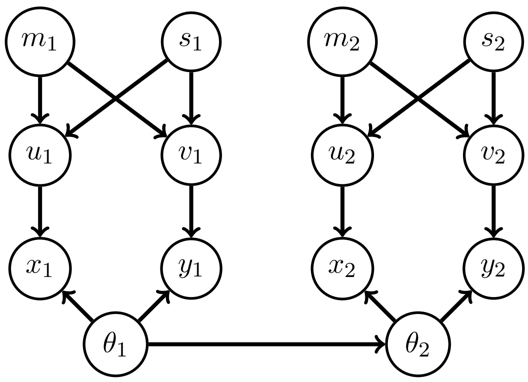
(d) $x_2 \perp\!\!\!\perp x_9 \mid \{x_6, x_8\}$

(e) $x_8 \perp\!\!\!\perp \{x_2, x_9\} \mid \{x_3, x_5, x_6, x_7\}$

(f) $\mathbb{E}[x_2 \cdot x_3 \cdot x_4 \cdot x_5 \cdot x_8 \mid x_7] = 0$ if $\mathbb{E}[x_8 \mid x_7] = 0$

Exercise 11. ***Independencies in directed graphical models***

Consider the following directed acyclic graph:



For each of the statements below, determine whether it holds for all probabilistic models that factorise over the graph. Provide a justification for your answer.

(a) $x_1 \perp\!\!\!\perp x_2$

(b) $p(x_1, y_1, \theta_1, u_1) \propto \phi_A(x_1, \theta_1, u_1)\phi_B(y_1, \theta_1, u_1)$ for some non-negative functions ϕ_A and ϕ_B

(c) $v_2 \perp\!\!\!\perp \{u_1, v_1, u_2, x_2\} \mid \{m_2, s_2, y_2, \theta_2\}$

(d) $\mathbb{E}[m_2 \mid m_1] = \mathbb{E}[m_2]$