Factor Graphs

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Recap

- Undirected and directed graphical models have complementary properties
- Both encode and (visually) represent statistical independencies (I-maps) and factorisations.
- For directed graphs with parent sets pa_i

$$p(x_1, \dots, x_d) = \prod_{i=1}^{d} p(x_i | \mathrm{pa}_i)$$

- For undirected graphs with maximal clique sets $\mathcal{X}_c$

$$p(x_1, \dots, x_d) = \frac{1}{Z} \prod_{c} \phi_c(\mathcal{X}_c)$$

Program

1. What are factor graphs?
2. Advantages over directed or undirected graphs?

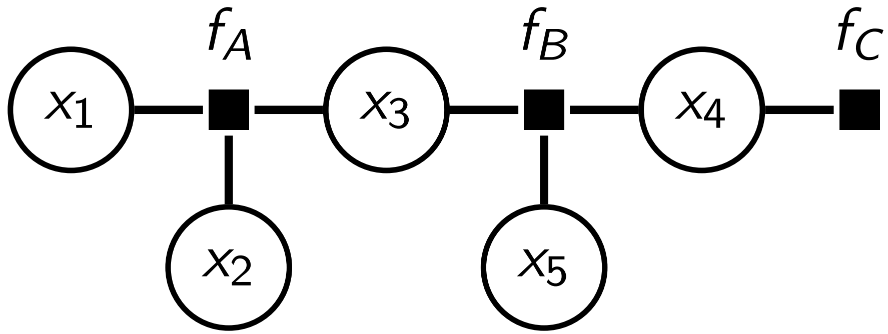
Program

1. What are factor graphs?
   - Definition
   - Visualising Gibbs distributions as factor graphs
   - Visualising factors that are conditionals
2. Advantages over directed or undirected graphs?

Definition of factor graphs

- A factor graph represents the factorisation of an arbitrary function (not necessarily related to pdfs/pmfs)
- Example: $h(x_1, \ldots, x_5) = f_A(x_1, x_2, x_3) f_B(x_3, x_4, x_5) f_C(x_4)$

  Factor graph (FG):



- Two types of nodes: factor and variable nodes
- Convention: squares for factors, circles for variables (other conventions are used too)

Definition of factor graphs

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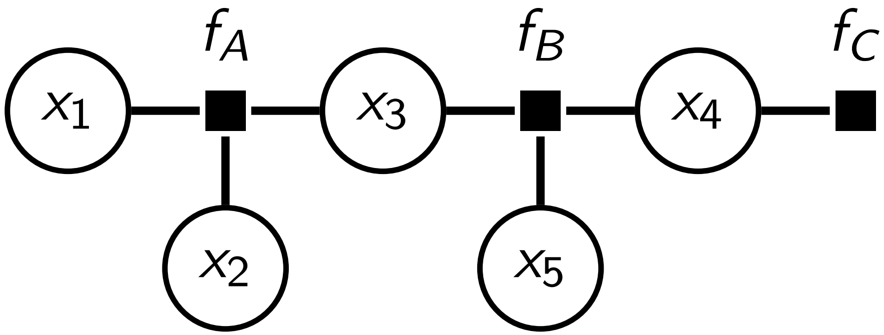
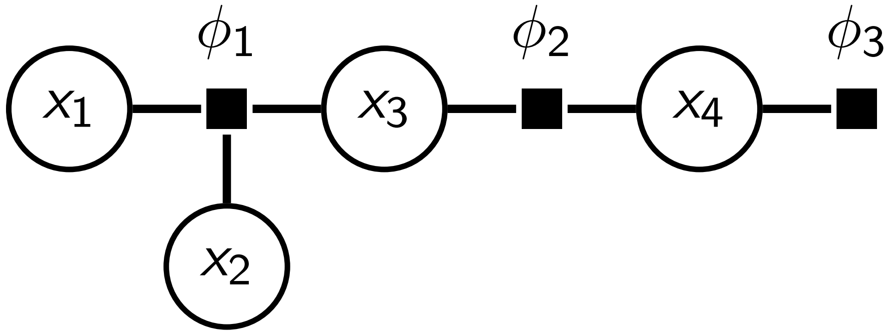
  Factor graph (FG):



- Edge between variable x and factor $f \Leftrightarrow x$ is an argument of f
- Variable nodes are always connected to factor nodes; no direct links between factor or variable nodes (FGs are bipartite graphs)
- We can also use directed edges (to indicate conditionals)

Visualising Gibbs distributions as factor graphs

- Example: $p(x_1, x_2, x_3, x_4) = \frac{1}{Z}\phi_1(x_1, x_2, x_3)\phi_2(x_3, x_4)\phi_3(x_4)$



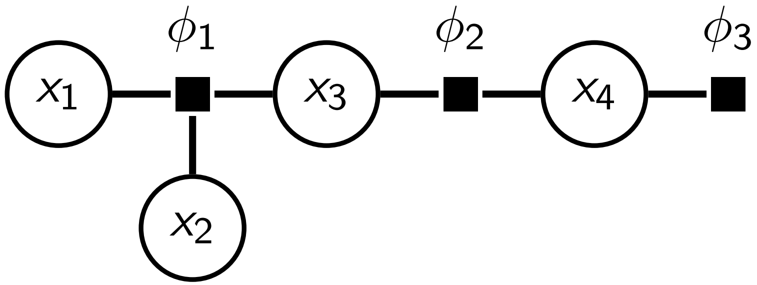
- General case: $p(x_1, \ldots, x_d) \propto \prod_c \phi_c(\mathcal{X}_c)$
  - Factor node for all ϕ_c
  - For all factors ϕ_c:
    draw an undirected edge between ϕ_c and all $x_i \in \mathcal{X}_c$.
- Can visualise any undirected graphical model as a factor graph.

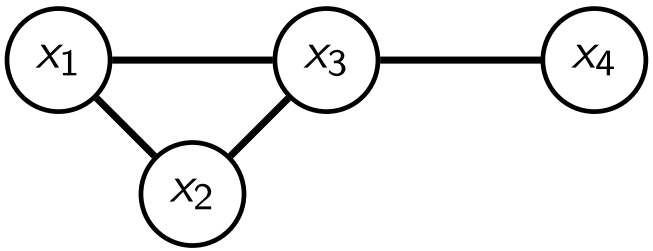
Visualising Gibbs distributions as factor graphs

Some differences to visualisation with undirected graph:

- Factors ϕ_c are shown, which makes the graphs more informative (see next slide).
- Variables x_i are neighbours if they are connected to the same factor.

$p(x_1, x_2, x_3, x_4) = \frac{1}{Z}\phi_1(x_1, x_2, x_3)\phi_2(x_3, x_4)\phi_3(x_4)$

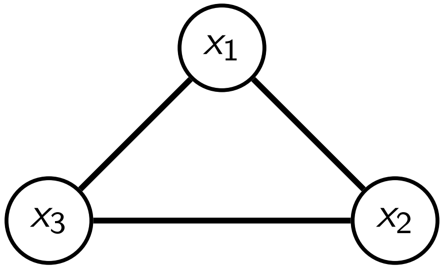




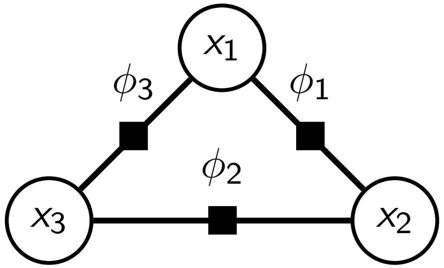
More informative than undirected graphs

- Mapping from Gibbs distribution to undirected graph is many to one but one-to-one for factor graphs.
- Example

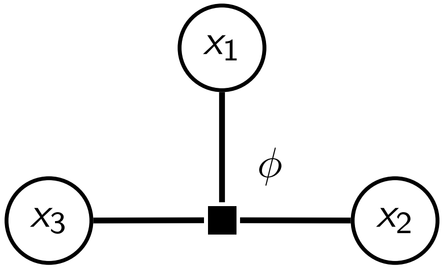
$$p_A(x_1, x_2, x_3) \propto \phi_1(x_1, x_2)\phi_2(x_2, x_3)\phi_3(x_3, x_1)$$
$$p_B(x_1, x_2, x_3) \propto \phi(x_1, x_2, x_3)$$



UG for p_A and p_B



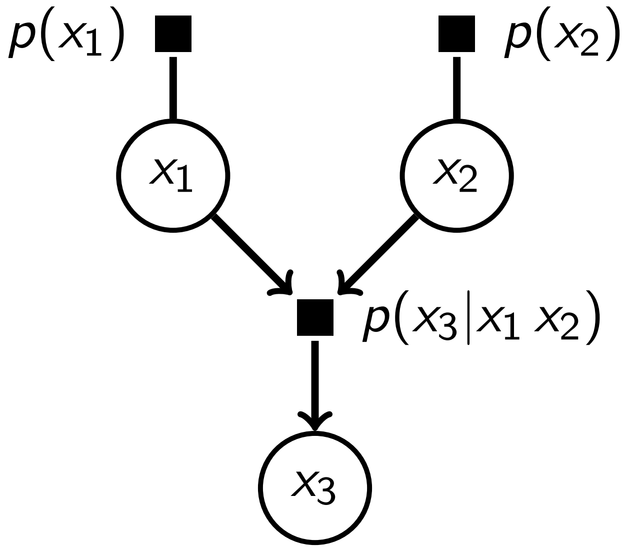
FG for p_A



FG for p_B

Visualising factors that are conditionals

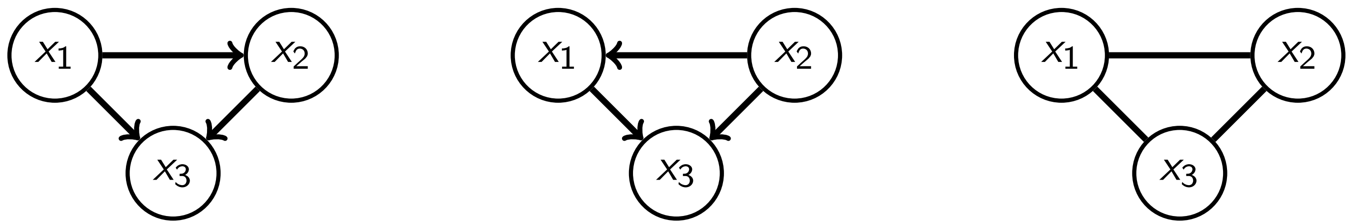
- For $p(x_1, x_2, x_3) = p(x_1)p(x_2)p(x_3|x_1, x_2)$, we may want to include the information that x_3 is conditioned on x_1, x_2
- Use arrows as in directed graphs.



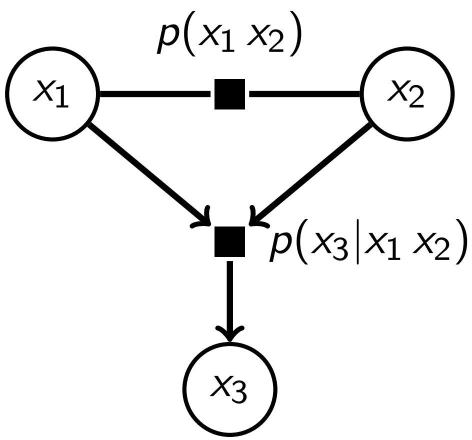
- Can visualise any directed graphical model as a factor graph.

Mixed graphs

- Let $p(x_1, x_2, x_3) = p(x_1, x_2)p(x_3|x_1, x_2)$.
- Directed graphs forces ordering of the random variables; undirected graph does not show conditioning on x_1, x_2



- Mixed FG to visualise the conditioning for $p(x_3|x_1, x_2)$ without imposing an ordering on x_1 and x_2



Program

1. What are factor graphs?

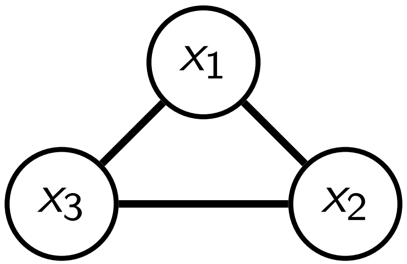
2. Advantages over directed or undirected graphs?
   - Computational advantages
   - Statistical advantages

Importance of factorisation

- Factorisation was central in the development so far
- But directed and undirected graphs are not able to fully represent arbitrary factorisations of pdfs/pmfs.

  For example, same graph for

$$p(x_1, x_2, x_3) \propto \phi_1(x_1, x_2)\phi_2(x_2, x_3)\phi_3(x_3, x_1)$$
$$p(x_1, x_2, x_3) \propto \phi(x_1, x_2, x_3)$$

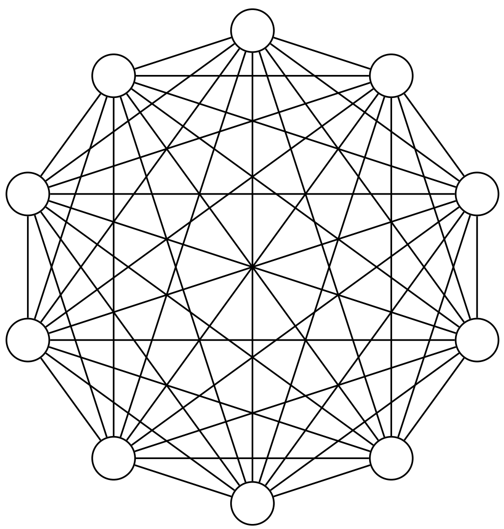


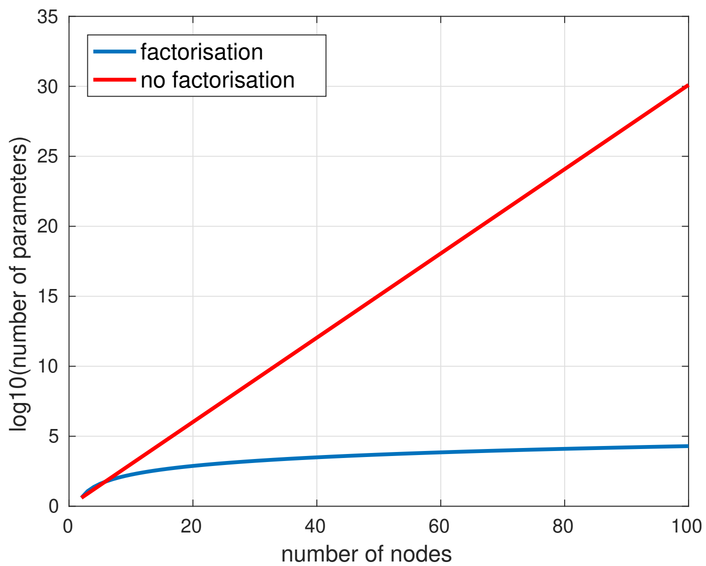
- We should expect that being able to better represent the factorisation has advantages.

Example of computational advantages

Assume binary random variables x_i.

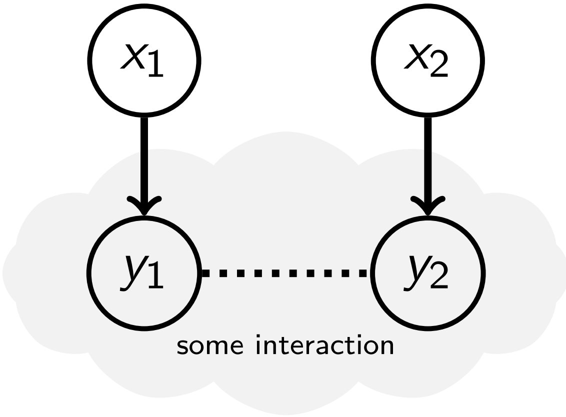
- Same undirected graph but
  $p(x_1, \ldots, x_d) \propto \phi(x_1, \ldots, x_d)$ has 2^d free parameters,
  $p(x_1, \ldots, x_d) \propto \prod_{i<j} \phi_{ij}(x_i, x_j)$ has $\binom{d}{2} 2^2$ free parameters
  parameters $\equiv$ entries to specify in a table representation
- The difference matters for learning and inference when the number of variables is large.





Example of statistical advantages

- Let x_1 and x_2 be two inputs
- x_1 controls variable y_1
  x_2 controls y_2
- Variables y_1 and y_2 influence each other



- Model: $p(y_1, y_2, x_1, x_2) = p(y_1, y_2 | x_1, x_2)p(x_1)p(x_2)$
  (probabilistic modelling: pdf/pmf $p(y_1, y_2|x_1, x_2)$ captures uncertainty about how the x_i affect the y_i and about how the y_i interact)

- Choose $p(y_1, y_2|x_1, x_2)$ such that $p(y_1, y_2, x_1, x_2)$ satisfies
  - $x_1 \perp\!\!\!\perp x_2$ (independence between control variables)
  - $x_1 \perp\!\!\!\perp y_2 \mid y_1, x_2$ (y_2 is only directly influenced by y_1 and x_2)
  - $x_2 \perp\!\!\!\perp y_1 \mid y_2, x_1$ (y_1 is only directly influenced by y_2 and x_1)

Example of statistical advantages

- Three independencies are satisfied if $p(y_1, y_2|x_1, x_2)$ factorises as

$$p(y_1, y_2|x_1, x_2) = p(y_1|x_1)p(y_2|x_2)\phi(y_1, y_2)n(x_1, x_2)$$

  where $n(x_1, x_2)$ ensures normalisation of $p(y_1, y_2|x_1, x_2)$

$$n(x_1, x_2) = \left(\int p(y_1|x_1)p(y_2|x_2)\phi(y_1, y_2)\mathrm{d}y_1\mathrm{d}y_2\right)^{-1}$$

  (see exercises)
- Directed and undirected graphs cannot represent the independencies induced by factorisation of $p(y_1, y_2|x_1, x_2)$ (see exercises).
- Factor graphs and chain graphs (see e.g. Barber, Section 4.3; Lauritzen, Section 3.2.3, not covered in PMR) can represent them.
- Factor graphs can represent independencies that DAGs or UGs cannot or do not represent (not covered in PMR).

Program recap

1. What are factor graphs?
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   - Visualising factors that are conditionals

2. Advantages over directed or undirected graphs?
   - Computational advantages
   - Statistical advantages

Credits

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