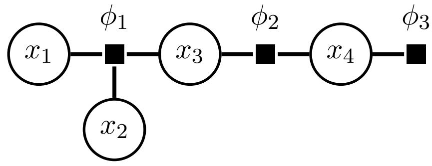
# Probabilistic Modelling and Reasoning
# Exercises 4 — Notes

Spring 2022
Hodari & Gutmann

*These notes are intended to give a summary of relevant concepts from the lectures which are helpful to complete the exercises. It is not intended to cover the lectures thoroughly. Learning this content is not a replacement for working through the lecture material and the exercises.*

**Factor graph —** A factor graph represents an arbitrary function in terms of factors and their connections with variables. For example, a factor graph can represent a distribution written as a Gibbs distribution – $p(\mathbf{x}) = \frac{1}{Z}\prod_c \phi_c(\mathcal{X}_c)$ – where variables $x_i \in \mathbf{x}$ are represented with variable nodes (circles) and potentials $\phi_c$ are represented with factor nodes (squares). Edges connect each factor node $\phi_c$ to all its variable nodes $x_i \in \mathcal{X}_c$.

$$p(x_1, x_2, x_3, x_4) = \frac{1}{Z}\phi_1(x_1, x_2, x_3)\phi_2(x_3, x_4)\phi_3(x_4)$$




**Variable elimination —** Given $p(\mathcal{X}) \propto \prod_c \phi_c(\mathcal{X}_c)$, we compute the marginal $p(\mathcal{X} \setminus x^*)$ via the sum rule by exploiting the factorisation by means of the distributive law.

We sum out the variable $x^*$ by first finding all factors $\phi_i(\mathcal{X}_i)$ such that $x^* \in \mathcal{X}_i$, and forming the compound factor $\phi^*(\mathcal{X}^*) = \prod_{i:x^* \in \mathcal{X}_i} \phi_i(\mathcal{X}_i)$, with $\mathcal{X}^* = \bigcup_{i:x^* \in \mathcal{X}_i} \mathcal{X}_i$. Summing out $x^*$ then produces a new factor $\tilde{\phi}^*(\tilde{\mathcal{X}}^*) = \sum_{x^*} \phi^*(\mathcal{X}^*)$ that does not depend on $x^*$, i.e. $\tilde{\mathcal{X}}^* = \mathcal{X}^* \setminus x^*$. This is possible as products are commutative, and a sum can be distributed within a product as long as all terms depending on the variable(s) being summed come to the right of the sum.

$$p(\mathcal{X} \setminus x^*) \propto \sum_{x^*} \prod_c \phi_c(\mathcal{X}_c) \propto \left[\prod_{i:x^* \notin \mathcal{X}_i} \phi_i(\mathcal{X}_i)\right] \left[\sum_{x^*} \prod_{i:x^* \in \mathcal{X}_i} \phi_i(\mathcal{X}_i)\right] \tag{1}$$

$$\propto \left[\prod_{i:x^* \notin \mathcal{X}_i} \phi_i(\mathcal{X}_i)\right] \tilde{\phi}^*(\tilde{\mathcal{X}}^*) \tag{2}$$

When eliminating variables, order of elimination matters. However, optimal choice of elimination order is difficult. Picking variables greedily is a common heuristic, where the "best" $x^*$ is the one that fewest factors $\phi_c$ depend upon.

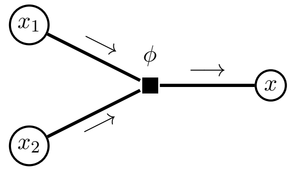
**Sum-product algorithm —** Variable elimination for factor trees reformulated with "messages" which allows for re-use of computations already done. See table on following page.

**Max-product algorithm —** Same as the sum-product algorithm, but max replaces $\sum$.

**Max-sum algorithm —** Max-product algorithm in the log-domain. See table on following page.

**Sum-product algorithm**

$\mu_{\phi\to x}(x)$ Factor to variable
$\mu_{\phi\to x}(x) = \sum_{x_1,\dots,x_j} \phi(x_1,\dots,x_j,x) \prod_{i=1}^{j} \mu_{x_i\to\phi}(x_i)$
where $\{x_1,\dots,x_j\} = \mathrm{ne}(\phi) \setminus \{x\}$

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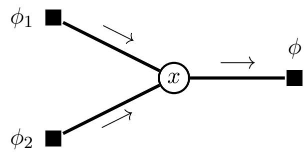
$\mu_{x\to\phi}(x)$ Variable to factor
$\mu_{x\to\phi}(x) = \prod_{i=1}^{j} \mu_{\phi_i\to x}(x)$
where $\{\phi_1,\dots,\phi_j\} = \mathrm{ne}(x) \setminus \{\phi\}$

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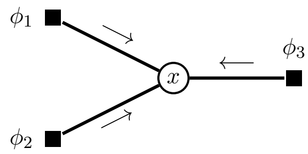
$\tilde{p}(x)$ Univariate marginals – unnormalised
$p(x) \propto \prod_{i=1}^{j} \mu_{\phi_i\to x}(x)$
where $\{\phi_1,\dots,\phi_j\} = \mathrm{ne}(x)$

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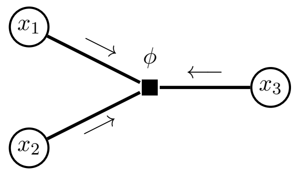
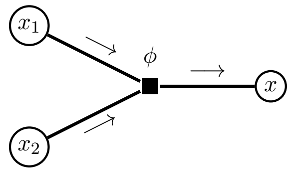
$\tilde{p}(x_1,\dots,x_j)$ Joint marginals of variables sharing a factor– unnormalised
$p(x_1,\dots,x_j) \propto \phi(x_1,\dots,x_j) \prod_{i=1}^{j} \mu_{x_i\to\phi}(x_i)$
where $\{x_1,\dots,x_j\} = \mathrm{ne}(\phi)$




**Max-sum algorithm**

$\gamma_{\phi\to x}(x)$ Factor to variable
$\gamma_{\phi\to x}(x) = \max_{x_1,\dots,x_j} \log\phi(x_1,\dots,x_j,x) + \sum_{i=1}^{j} \gamma_{x_i\to\phi}(x_i)$
$\gamma^*_{\phi\to x}(x) = \mathrm{argmax}_{x_1,\dots,x_j} \log\phi(x_1,\dots,x_j,x) + \sum_{i=1}^{j} \gamma_{x_i\to\phi}(x_i)$
where $\{x_1,\dots,x_j\} = \mathrm{ne}(\phi) \setminus \{x\}$

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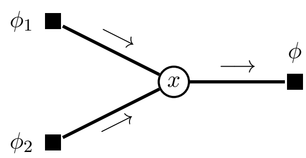
$\gamma_{x\to\phi}(x)$ Variable to factor
$\gamma_{x\to\phi}(x) = \sum_{i=1}^{j} \gamma_{\phi_i\to x}(x)$
where $\{\phi_1,\dots,\phi_j\} = \mathrm{ne}(x) \setminus \{\phi\}$

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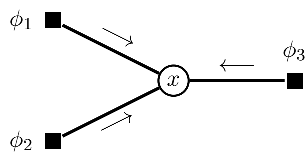
$\log p_{\max}$ Maximum probability
$\log p_{\max} = \max_x \gamma^*(x), \quad \gamma^*(x) = -\log Z + \sum_{i=1}^{j} \gamma_{\phi_i\to x}(x)$
where $\{\phi_1,\dots,\phi_j\} = \mathrm{ne}(x)$

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$\mathrm{argmax}_{\mathbf{x}}\, \tilde{p}(\mathbf{x})$ Maximum probability states – no need for normalisation
Init: $\hat{x} = \mathrm{argmax}_x \gamma^*(x) = \mathrm{argmax}_x \sum_{i=1}^{j} \gamma_{\phi_i\to x}(x)$
Backtrack to leaves via $\gamma^*_{\phi\to x}(x)$