Probabilistic Modelling and Reasoning
— Introduction —

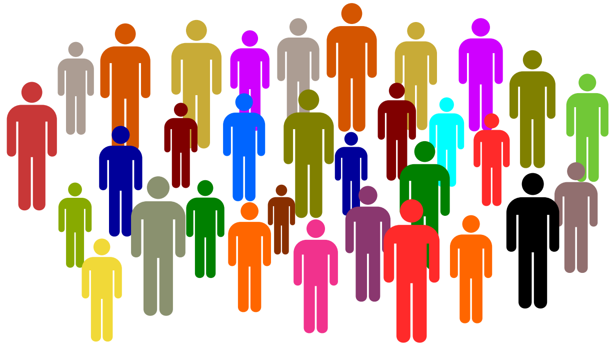
Michael U. Gutmann

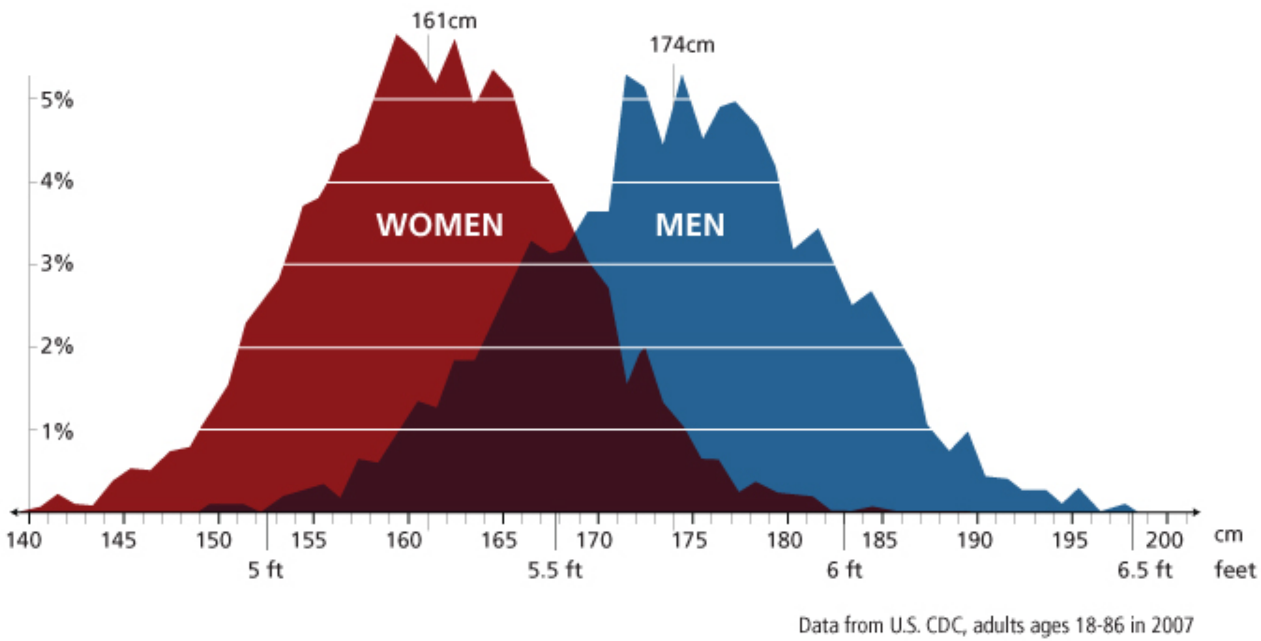
Probabilistic Modelling and Reasoning (INFR11134)
School of Informatics, The University of Edinburgh

Spring Semester 2022

Variability

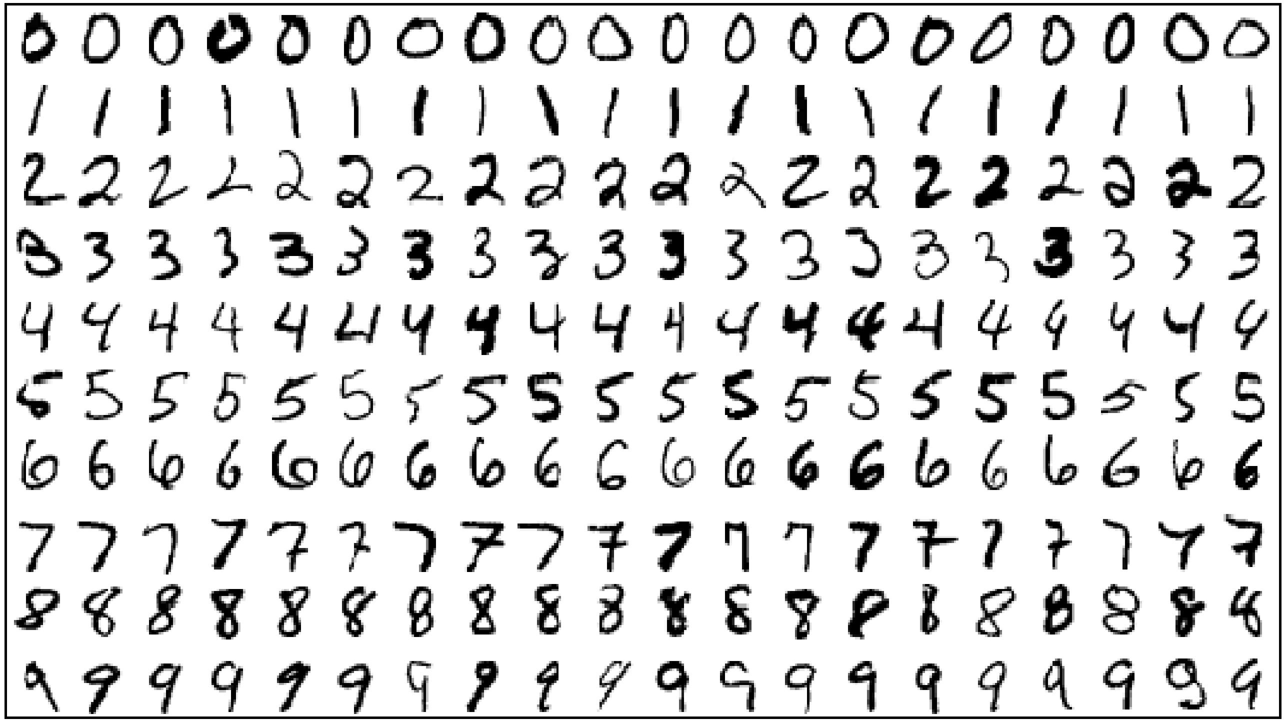
- Variability is part of nature
- Human heights vary
- Men are typically taller than women but height varies a lot





Variability

- Our handwriting is unique
- Variability leads to uncertainty: e.g. 1 vs 7 or 4 vs 9



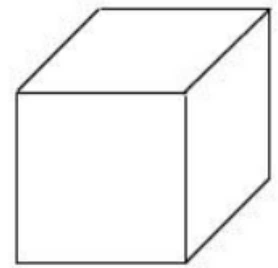
- Variability leads to uncertainty
- Reading handwritten text in a foreign language

Example: Screening and diagnostic tests

- Early warning test for Alzheimer's disease (Scharre, 2010, 2014)
- Detects "mild cognitive impairment"

- Takes 10–15 minutes
- Freely available
- Assume a 70 year old man tests positive.
- Should he be concerned?

7. Copy this picture:



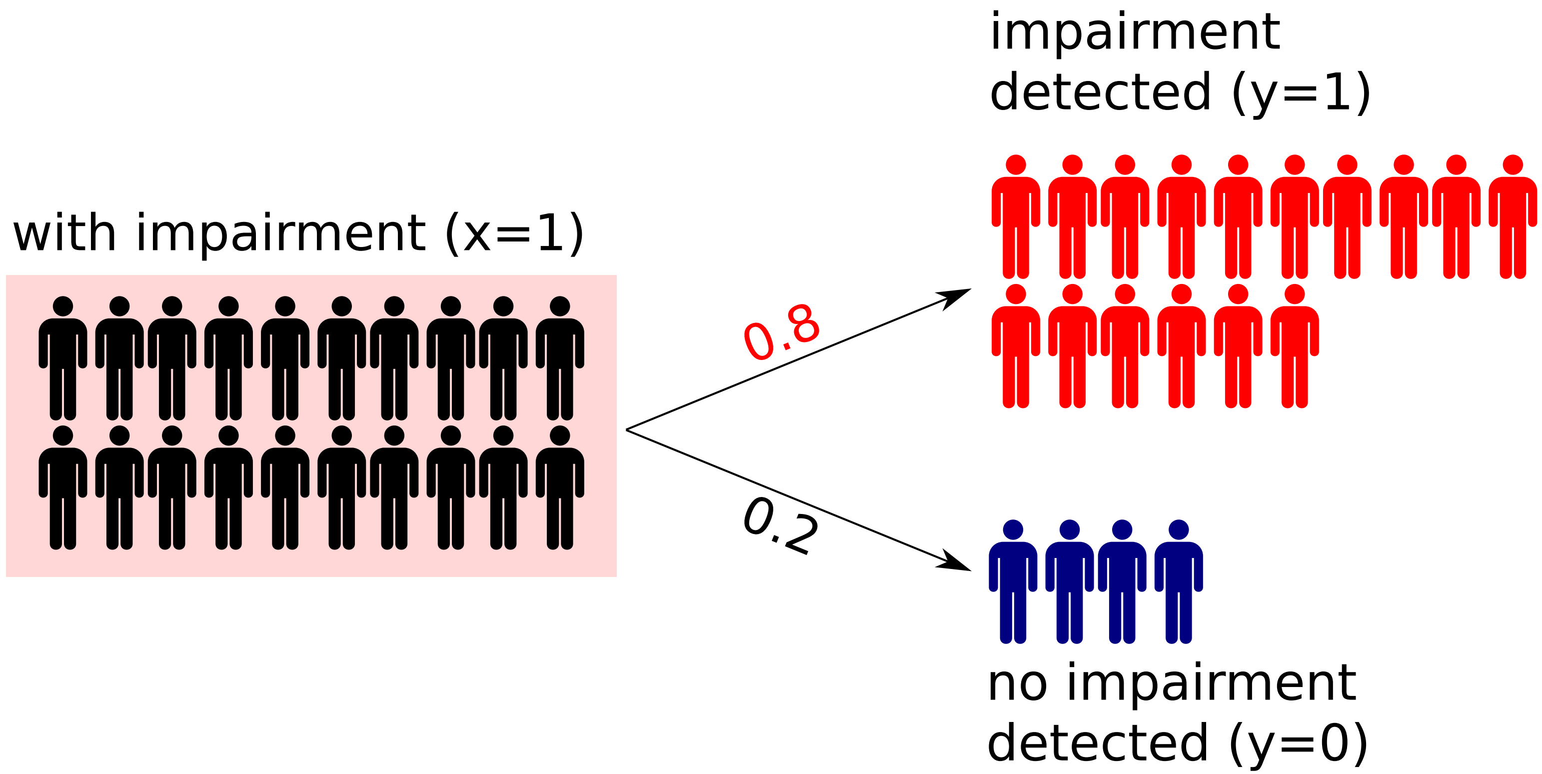
8. Drawing test

- Draw a large face of a clock and place in the numbers
- Position the hands for 5 minutes after 11 o'clock

(Example from sagetest.osu.edu)

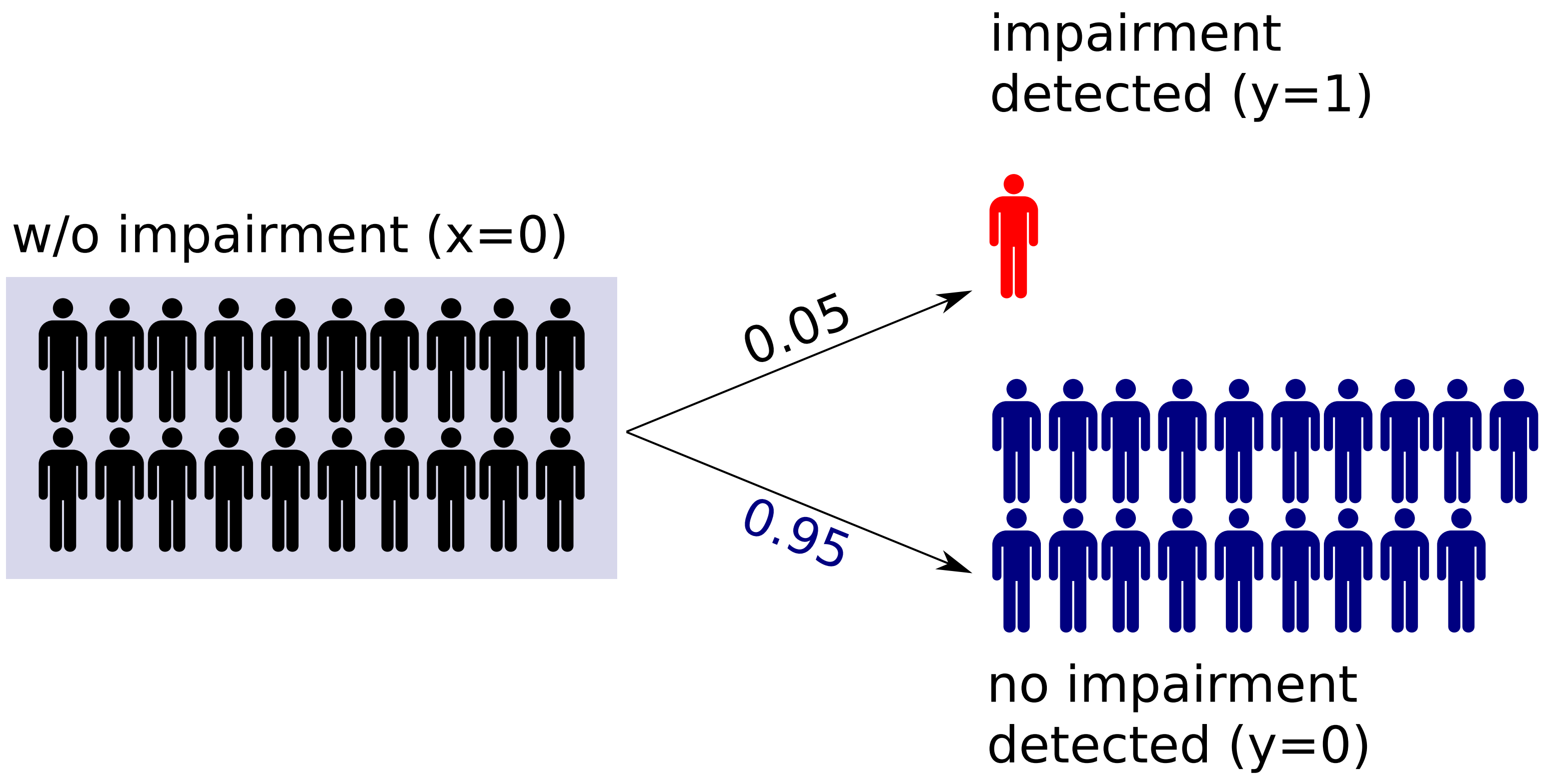
Accuracy of the test

- Sensitivity of 0.8 and specificity of 0.95 (Scharre, 2010)
- 80% correct for people with impairment



Accuracy of the test

- Sensitivity of 0.8 and specificity of 0.95 (Scharre, 2010)
- 95% correct for people w/o impairment



Variability implies uncertainty

- People of the same group do not have the same test results
  - Test outcome is subject to variability
  - The data are noisy
- Variability leads to uncertainty
  - Positive test $\equiv$ true positive ?
  - Positive test $\equiv$ false positive ?
- What can we safely conclude from a positive test result?
- How should we analyse such kind of ambiguous data?

Probabilistic approach

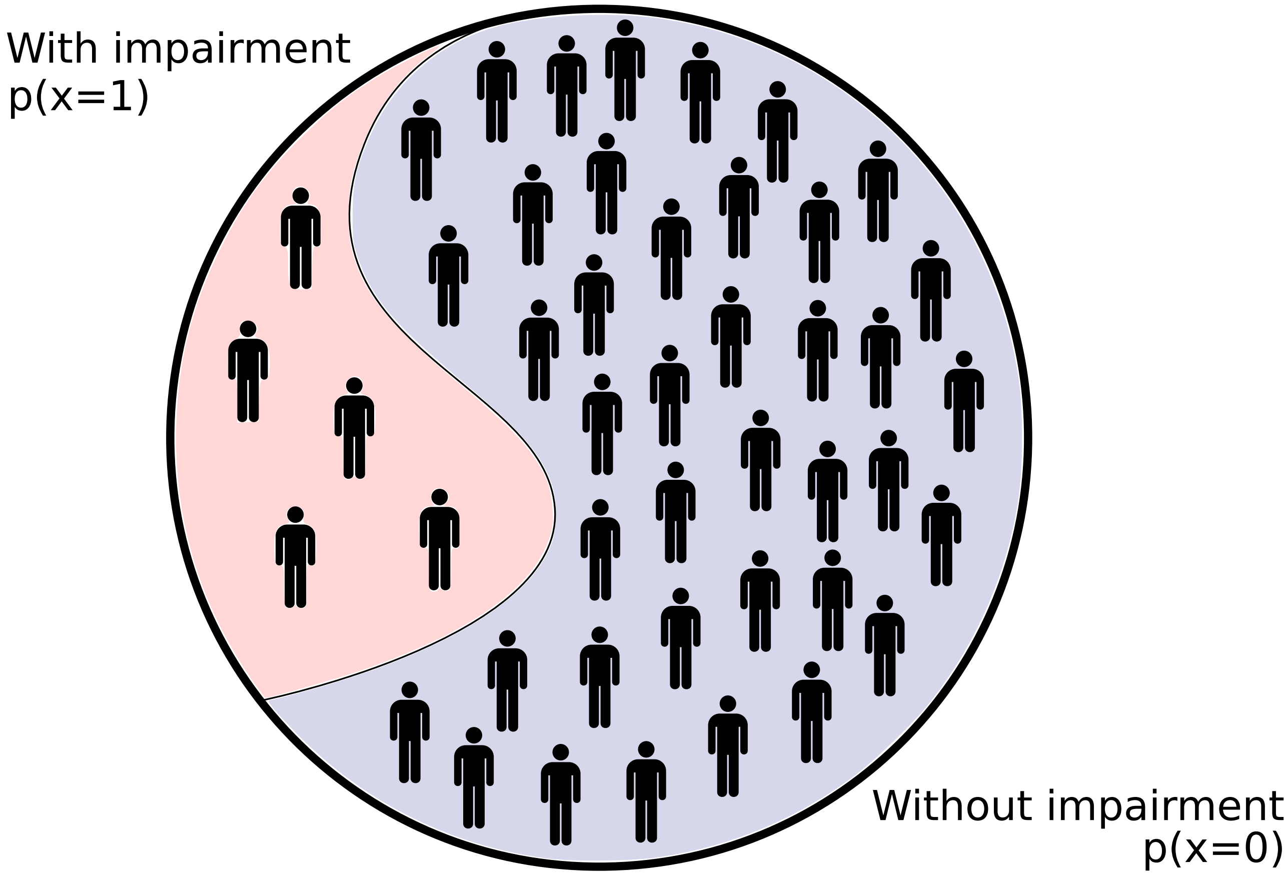
- The test outcomes y can be described with probabilities

$$
\begin{aligned}
\text{sensitivity} = 0.8 \quad &\Leftrightarrow \quad \mathbb{P}(y = 1|x = 1) = 0.8 \\
&\Leftrightarrow \quad \mathbb{P}(y = 0|x = 1) = 0.2 \\
\text{specificity} = 0.95 \quad &\Leftrightarrow \quad \mathbb{P}(y = 0|x = 0) = 0.95 \\
&\Leftrightarrow \quad \mathbb{P}(y = 1|x = 0) = 0.05
\end{aligned}
$$

- $\mathbb{P}(y|x)$: model of the test specified in terms of (conditional) probabilities
- $x \in \{0, 1\}$: quantity of interest (cognitive impairment or not)

Prior information

Among people like the patient, $\mathbb{P}(x = 1) = 5/45 \approx 11\%$ have a cognitive impairment (plausible range: 3% – 22%, Geda, 2014)

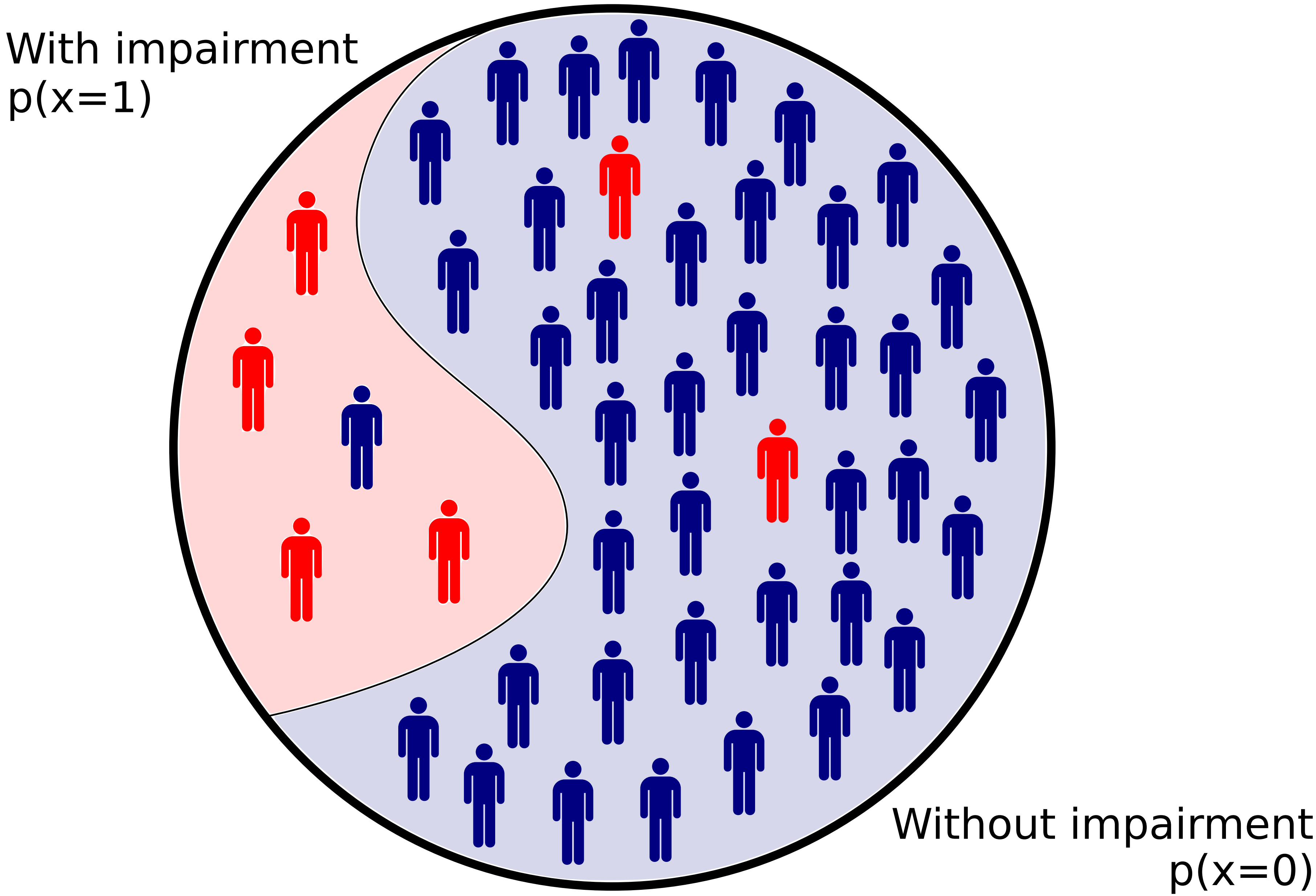


Probabilistic model

- Reality:
  - properties/characteristics of the group of people like the patient
  - properties/characteristics of the test
- Probabilistic model:
  - $\mathbb{P}(x = 1)$
  - $\mathbb{P}(y = 1|x = 1)$ or $\mathbb{P}(y = 0|x = 1)$
    $\mathbb{P}(y = 1|x = 0)$ or $\mathbb{P}(y = 0|x = 0)$

  Fully specified by three numbers.
- A probabilistic model is an abstraction of reality that uses probability theory to quantify the chance of uncertain events.

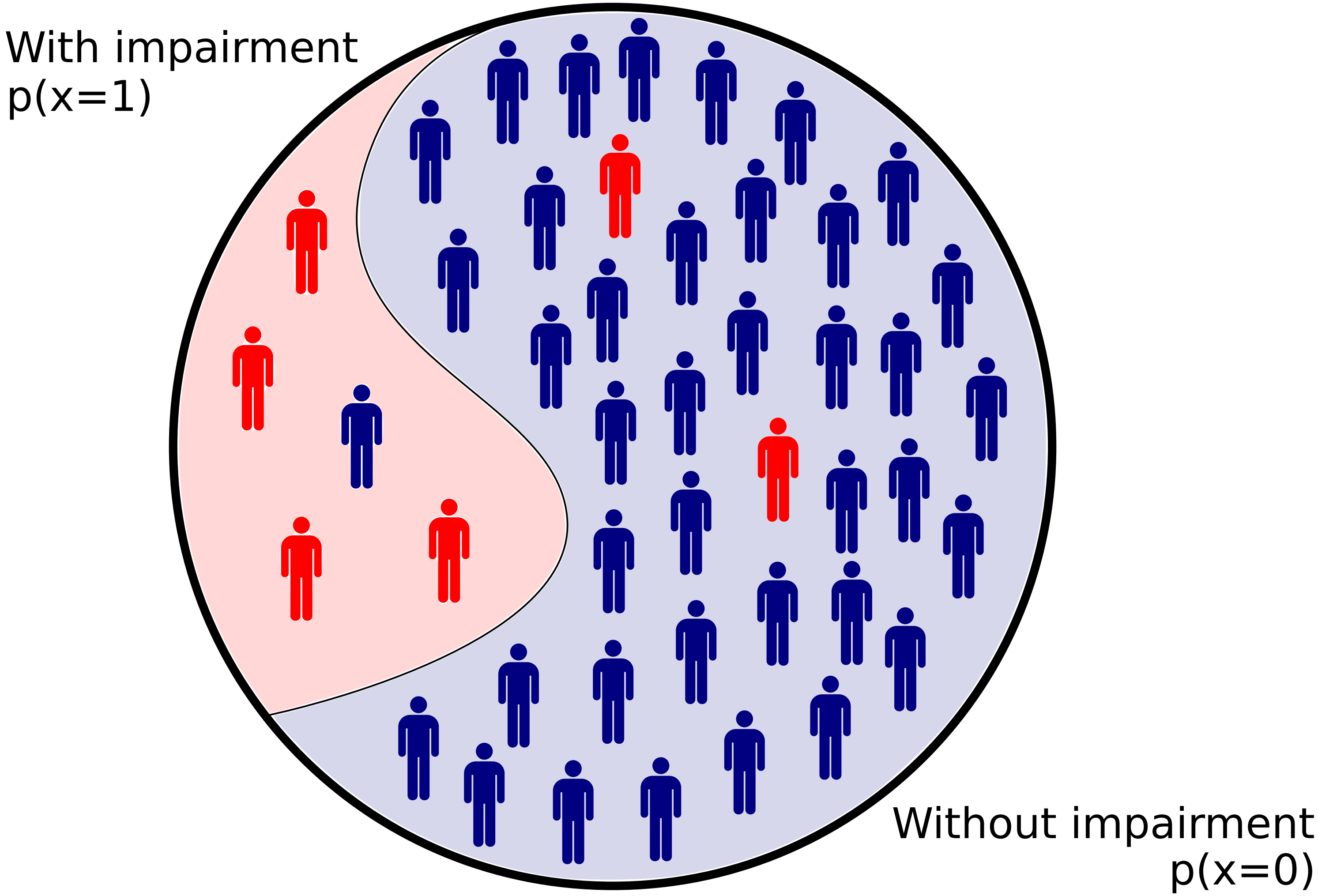
If we tested the whole population



If we tested the whole population

Fraction of people who are impaired and have positive tests:

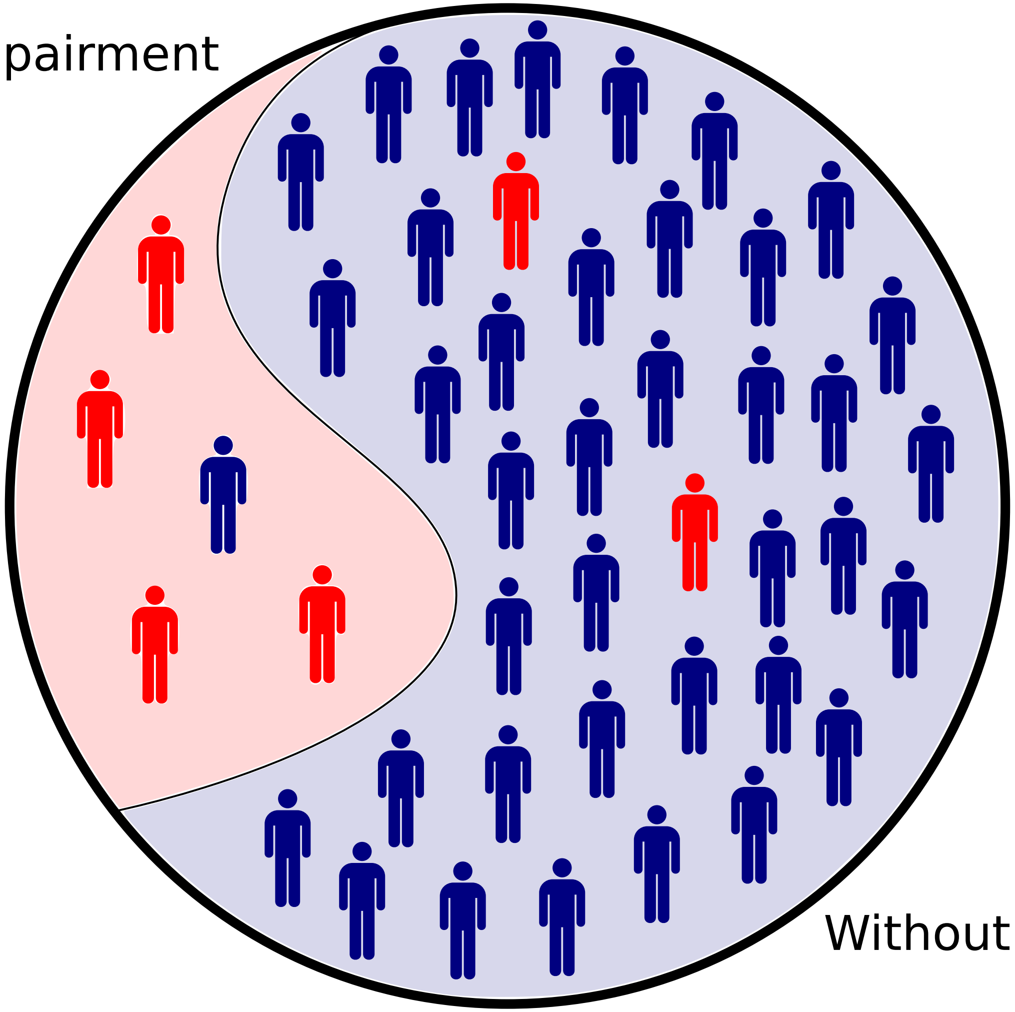
$$\mathbb{P}(x = 1, y = 1) = \mathbb{P}(y = 1|x = 1)\mathbb{P}(x = 1) = 4/45 \qquad \text{(product rule)}$$



If we tested the whole population

Fraction of people who are not impaired but have positive tests:

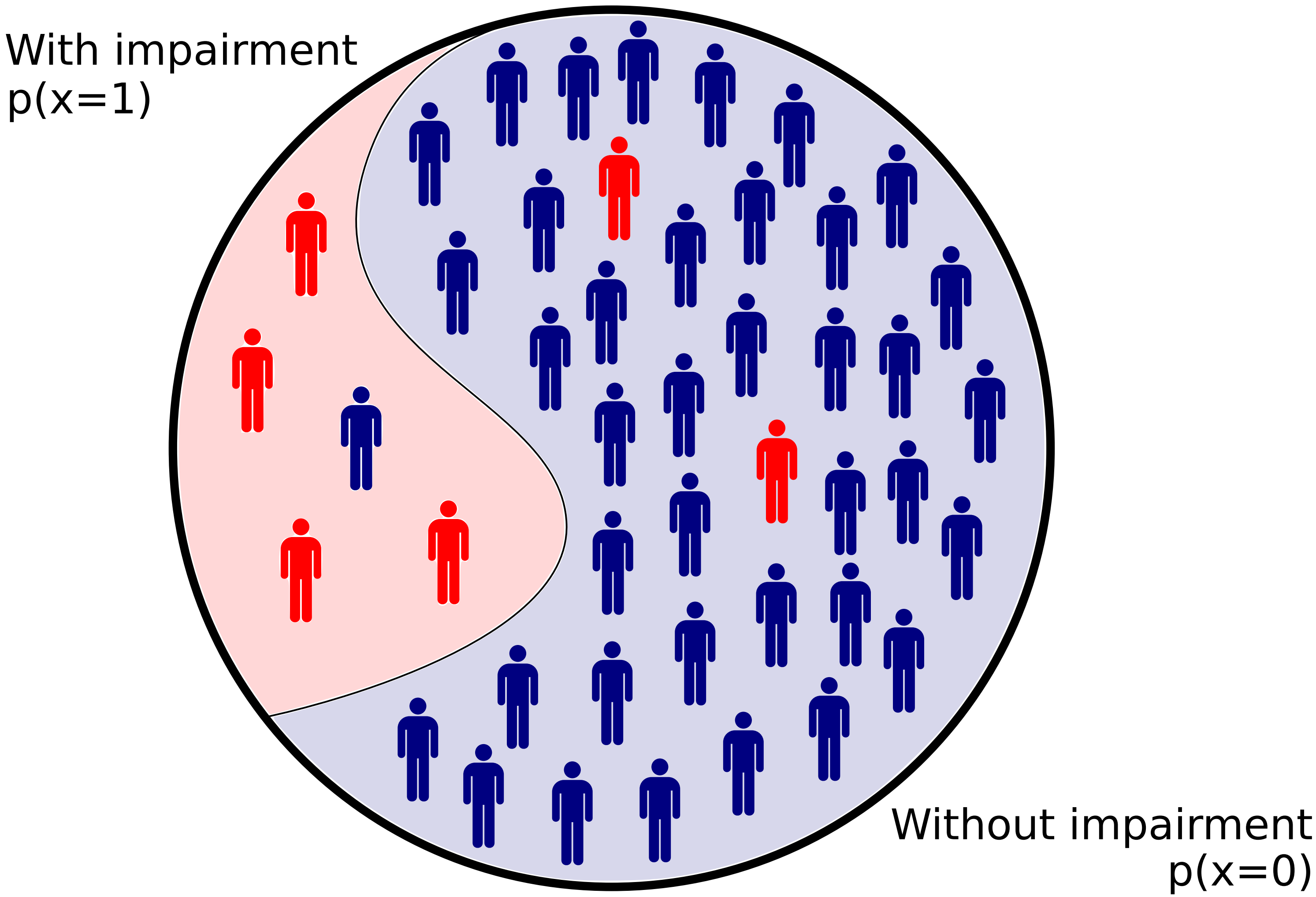
$$\mathbb{P}(x = 0, y = 1) = \mathbb{P}(y = 1|x = 0)\mathbb{P}(x = 0) = 2/45 \qquad \text{(product rule)}$$



If we tested the whole population

Fraction of people where the test is positive:

$$\mathbb{P}(y=1) = \mathbb{P}(x=1, y=1) + \mathbb{P}(x=0, y=1) = 6/45 \qquad \text{(sum rule)}$$



Putting everything together

- Among those with a positive test, fraction with impairment:

$$\mathbb{P}(x = 1|y = 1) = \frac{\mathbb{P}(y = 1|x = 1)\mathbb{P}(x = 1)}{\mathbb{P}(y = 1)} = \frac{4}{6} = \frac{2}{3}$$

- Fraction without impairment:

$$\mathbb{P}(x = 0|y = 1) = \frac{\mathbb{P}(y = 1|x = 0)\mathbb{P}(x = 0)}{\mathbb{P}(y = 1)} = \frac{2}{6} = \frac{1}{3}$$

- Equations are examples of "Bayes' rule".
- Positive test increased probability of cognitive impairment from 11% (prior belief) to 67%, or from 6% to 51%.
- 51% $\approx$ coin flip

Probabilistic reasoning

- Probabilistic reasoning $\equiv$ probabilistic inference: Computing the probability of an event that we have not or cannot observe from an event that we can observe
  - Unobserved/uncertain event, e.g. cognitive impairment $x = 1$
  - Observed event $\equiv$ evidence $\equiv$ data, e.g. test result $y = 1$
- "The prior": probability for the uncertain event before having seen evidence, e.g. $\mathbb{P}(x = 1)$
- "The posterior": probability for the uncertain event after having seen evidence, e.g. $\mathbb{P}(x = 1|y = 1)$
- The posterior is computed from the prior and the evidence via Bayes' rule.

Key rules of probability

(1) Product rule:

$$\begin{aligned}\mathbb{P}(x = 1, y = 1) &= \mathbb{P}(y = 1|x = 1)\mathbb{P}(x = 1) \\ &= \mathbb{P}(x = 1|y = 1)\mathbb{P}(y = 1)\end{aligned}$$

(2) Sum rule:

$$\mathbb{P}(y = 1) = \mathbb{P}(x = 1, y = 1) + \mathbb{P}(x = 0, y = 1)$$

Bayes' rule (conditioning) as consequence of the product rule

$$\mathbb{P}(x = 1|y = 1) = \frac{\mathbb{P}(x = 1, y = 1)}{\mathbb{P}(y = 1)} = \frac{\mathbb{P}(y = 1|x = 1)\mathbb{P}(x = 1)}{\mathbb{P}(y = 1)}$$

Denominator from sum rule, or sum rule and product rule

$$\mathbb{P}(y = 1) = \mathbb{P}(y = 1|x = 1)\mathbb{P}(x = 1) + \mathbb{P}(y = 1|x = 0)\mathbb{P}(x = 0)$$

Key rules or probability

- The rules generalise to the case of multivariate random variables (discrete or continuous)
- Consider the conditional joint probability density function (pdf) or probability mass function (pmf) of $\mathbf{x}, \mathbf{y}$: $p(\mathbf{x}, \mathbf{y})$

(1) Product rule:

$$\begin{aligned} p(\mathbf{x}, \mathbf{y}) &= p(\mathbf{x}|\mathbf{y})p(\mathbf{y}) \\ &= p(\mathbf{y}|\mathbf{x})p(\mathbf{x}) \end{aligned}$$

(2) Sum rule:

$$p(\mathbf{y}) = \begin{cases} \sum_{\mathbf{x}} p(\mathbf{x}, \mathbf{y}) & \text{for discrete r.v.} \\ \int p(\mathbf{x}, \mathbf{y}) \mathrm{d}\mathbf{x} & \text{for continuous r.v.} \end{cases}$$

Probabilistic modelling and reasoning

- Probabilistic modelling:
  - Identify the quantities that relate to the aspects of reality that you wish to capture with your model.
  - Consider them to be random variables, e.g. $\mathbf{x}, \mathbf{y}, \mathbf{z}$, with a joint pdf (pmf) $p(\mathbf{x}, \mathbf{y}, \mathbf{z})$.
- Probabilistic reasoning:
  - Assume you know that $\mathbf{y} \in \mathcal{E}$ (measurement, evidence)
  - Probabilistic reasoning about $\mathbf{x}$ then consists in computing

    $$p(\mathbf{x}|\mathbf{y} \in \mathcal{E})$$

    or related quantities like $\mathrm{argmax}_{\mathbf{x}}\, p(\mathbf{x}|\mathbf{y} \in \mathcal{E})$ or posterior expectations of some function g of $\mathbf{x}$, e.g.

    $$\mathbb{E}\left[g(\mathbf{x}) \mid \mathbf{y} \in \mathcal{E}\right] = \int g(\mathbf{u}) p(\mathbf{u}|\mathbf{y} \in \mathcal{E}) \mathrm{d}\mathbf{u}$$

Solution via product and sum rule

Assume that all variables are discrete valued, that $\mathcal{E} = \{\mathbf{y}_o\}$, and that we know $p(\mathbf{x}, \mathbf{y}, \mathbf{z})$. We would like to know $p(\mathbf{x}|\mathbf{y}_o)$.

- Product rule: $p(\mathbf{x}|\mathbf{y}_o) = \frac{p(\mathbf{x},\mathbf{y}_o)}{p(\mathbf{y}_o)}$
- Sum rule: $p(\mathbf{x}, \mathbf{y}_o) = \sum_{\mathbf{z}} p(\mathbf{x}, \mathbf{y}_o, \mathbf{z})$
- Sum rule: $p(\mathbf{y}_o) = \sum_{\mathbf{x}} p(\mathbf{x}, \mathbf{y}_o) = \sum_{\mathbf{x},\mathbf{z}} p(\mathbf{x}, \mathbf{y}_o, \mathbf{z})$
- Result:

$$p(\mathbf{x}|\mathbf{y}_o) = \frac{\sum_{\mathbf{z}} p(\mathbf{x}, \mathbf{y}_o, \mathbf{z})}{\sum_{\mathbf{x},\mathbf{z}} p(\mathbf{x}, \mathbf{y}_o, \mathbf{z})}$$

What we do in PMR

$$p(\mathbf{x}|\mathbf{y}_o) = \frac{\sum_{\mathbf{z}} p(\mathbf{x},\mathbf{y}_o,\mathbf{z})}{\sum_{\mathbf{x},\mathbf{z}} p(\mathbf{x},\mathbf{y}_o,\mathbf{z})}$$

Assume that $\mathbf{x}, \mathbf{y}, \mathbf{z}$ each are $d = 500$ dimensional, and that each element of the vectors can take $K = 10$ values.

- Issue 1: To specify $p(\mathbf{x}, \mathbf{y}, \mathbf{z})$, we need to specify $K^{3d} - 1 = 10^{1500} - 1$ non-negative numbers, which is impossible.

  Topic 1: Representation What reasonably weak assumptions can we make to efficiently represent $p(\mathbf{x}, \mathbf{y}, \mathbf{z})$?

What we do in PMR

$$p(\mathbf{x}|\mathbf{y}_o) = \frac{\sum_{\mathbf{z}} p(\mathbf{x}, \mathbf{y}_o, \mathbf{z})}{\sum_{\mathbf{x},\mathbf{z}} p(\mathbf{x}, \mathbf{y}_o, \mathbf{z})}$$

- Issue 2: The sum in the numerator goes over the order of $K^d = 10^{500}$ non-negative numbers and the sum in the denominator over the order of $K^{2d} = 10^{1000}$, which is impossible to compute.

  Topic 2: Exact inference Can we further exploit the assumptions on $p(\mathbf{x}, \mathbf{y}, \mathbf{z})$ to efficiently compute the posterior probability or derived quantities?

- Issue 3: Where do the non-negative numbers $p(\mathbf{x}, \mathbf{y}, \mathbf{z})$ come from?

  Topic 3: Learning How can we learn the numbers from data?

- Issue 4: For some models, exact inference and learning is too costly even after fully exploiting the assumptions made.

  Topic 4: Approximate inference and learning