The purpose of this additional sheet is to provide more practice and exam preparation material. N.B. The tutors are not required to work through this material in the tutorial.

Exercise 1. ***Maximum likelihood estimation for a Gaussian***

The Gaussian pdf parametrised by mean μ and standard deviation σ is given by

$$p(x; \boldsymbol{\theta}) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right], \qquad \boldsymbol{\theta} = (\mu, \sigma).$$

(a) Given iid data $\mathcal{D} = \{x_1, \ldots, x_n\}$, what is the likelihood function $L(\boldsymbol{\theta})$ for the Gaussian model?

(b) What is the log-likelihood function $\ell(\boldsymbol{\theta})$?

(c) Show that the maximum likelihood estimates for the mean μ and standard deviation σ are the sample mean

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \tag{1}$$

and the square root of the sample variance

$$S^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2. \tag{2}$$

Exercise 2. ***Posterior of the mean of a Gaussian with known variance***

Given iid data $\mathcal{D} = \{x_1, \ldots, x_n\}$, compute $p(\mu|\mathcal{D}, \sigma^2)$ for the Bayesian model

$$p(x|\mu) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] \qquad p(\mu; \mu_0, \sigma_0^2) = \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left[-\frac{(\mu-\mu_0)^2}{2\sigma_0^2}\right] \tag{3}$$

where σ^2 is a fixed known quantity.
Hint: You will need the result from Tutorial 5 for taking the product of Gaussians.