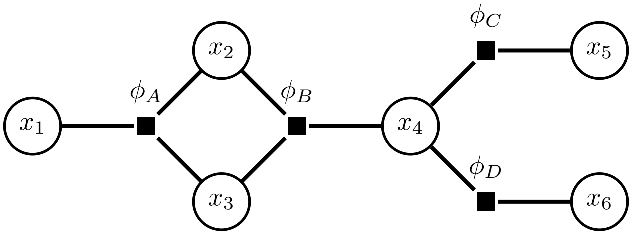
Exercise 1. ***Choice of elimination order in factor graphs***

Consider the following factor graph, which contains a loop:



Let all variables be binary, $x_i \in \{0, 1\}$, and the factors be defined as follows:

x_1	x_2	x_3	ϕ_A
0	0	0	4
1	0	0	2
0	1	0	2
1	1	0	6
0	0	1	2
1	0	1	6
0	1	1	6
1	1	1	4

x_2	x_3	x_4	ϕ_B
0	0	0	2
1	0	0	2
0	1	0	4
1	1	0	2
0	0	1	6
1	0	1	8
0	1	1	4
1	1	1	2

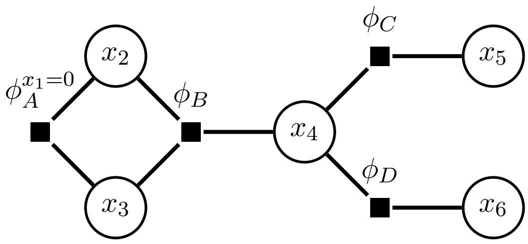
x_4	x_5	ϕ_C
0	0	8
1	0	2
0	1	2
1	1	6

x_4	x_6	ϕ_D
0	0	3
1	0	6
0	1	6
1	1	3

(a) *Draw the factor graph corresponding to $p(x_2, x_3, x_4, x_5 \mid x_1 = 0, x_6 = 1)$ and give the tables defining the new factors $\phi_A^{x_1=0}(x_2, x_3)$ and $\phi_D^{x_6=1}(x_4)$ that you obtain.*

Solution. First condition on $x_1 = 0$:

Factor node $\phi_A(x_1, x_2, x_3)$ depends on x_1, thus we create a new factor $\phi_A^{x_1=0}(x_2, x_3)$ from the table for ϕ_A using the rows where $x_1 = 0$.



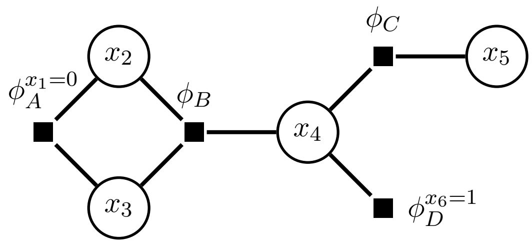
	x_1	x_2	x_3	ϕ_A
$\rightarrow$	0	0	0	4
	1	0	0	2
$\rightarrow$	0	1	0	2
	1	1	0	6
$\rightarrow$	0	0	1	2
	1	0	1	6
$\rightarrow$	0	1	1	6
	1	1	1	4

so that

x_2	x_3	$\phi_A^{x_1=0}$
0	0	4
1	0	2
0	1	2
1	1	6

Next condition on $x_6 = 1$:

Factor node $\phi_D(x_4, x_6)$ depends on x_6, thus we create a new factor $\phi_D^{x_6=1}(x_4)$ from the table for ϕ_D using the rows where $x_6 = 1$.



	x_4	x_6	ϕ_D
	0	0	3
	1	0	6
$\rightarrow$	0	1	6
$\rightarrow$	1	1	3

so that

x_4	$\phi_D^{x_6=1}$
0	6
1	3

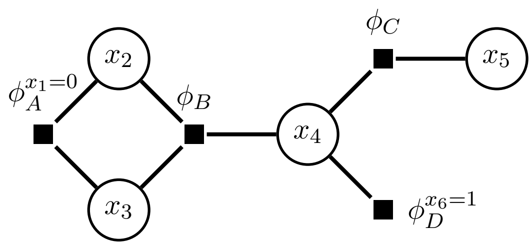
(b) *Find $p(x_2 \mid x_1 = 0, x_6 = 1)$ using the elimination ordering (x_4, x_5, x_3):*

(i) *Draw the graph for $p(x_2, x_3, x_5 \mid x_1 = 0, x_6 = 1)$ by marginalising x_4*
Compute the table for the new factor $\tilde{\phi}_4(x_2, x_3, x_5)$

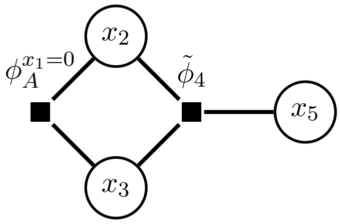
(ii) *Draw the graph for $p(x_2, x_3 \mid x_1 = 0, x_6 = 1)$ by marginalising x_5*
Compute the table for the new factor $\tilde{\phi}_{45}(x_2, x_3)$

(iii) *Draw the graph for $p(x_2 \mid x_1 = 0, x_6 = 1)$ by marginalising x_3*
Compute the table for the new factor $\tilde{\phi}_{453}(x_2)$

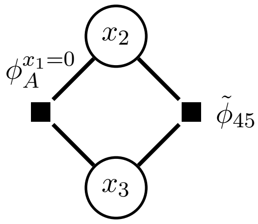
Solution. Starting with the factor graph for $p(x_2, x_3, x_4, x_5 \mid x_1 = 0, x_6 = 1)$



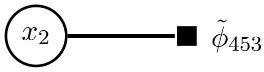
Marginalising x_4 combines the three factors ϕ_B, ϕ_C and $\phi_D^{x_6=1}$



Marginalising x_5 modifies the factor $\tilde{\phi}_4$



Marginalising x_3 combines the factors $\phi_A^{x_1=0}$ and $\tilde{\phi}_{45}$



We now compute the tables for the new factors $\tilde{\phi}_4$, $\tilde{\phi}_{45}$, $\tilde{\phi}_{453}$.
First find $\tilde{\phi}_4(x_2, x_3, x_5)$

x_2	x_3	x_4	ϕ_B
0	0	0	2
1	0	0	2
0	1	0	4
1	1	0	2
0	0	1	6
1	0	1	8
0	1	1	4
1	1	1	2

x_4	x_5	ϕ_C
0	0	8
1	0	2
0	1	2
1	1	6

x_4	$\phi_D^{x_6=1}$
0	6
1	3

so that

x_2	x_3	x_5	$\sum_{x_4} \phi_B(x_2, x_3, x_4)\phi_C(x_4, x_5)\phi_D^{x_6=1}(x_4)$		$\tilde{\phi}_4$
0	0	0	$(2 * 8 * 6) + (6 * 2 * 3)$	=	132
1	0	0	$(2 * 8 * 6) + (8 * 2 * 3)$	=	144
0	1	0	$(4 * 8 * 6) + (4 * 2 * 3)$	=	216
1	1	0	$(2 * 8 * 6) + (2 * 2 * 3)$	=	108
0	0	1	$(2 * 2 * 6) + (6 * 6 * 3)$	=	132
1	0	1	$(2 * 2 * 6) + (8 * 6 * 3)$	=	168
0	1	1	$(4 * 2 * 6) + (4 * 6 * 3)$	=	120
1	1	1	$(2 * 2 * 6) + (2 * 6 * 3)$	=	60

Next find $\tilde{\phi}_{45}(x_2, x_3)$

x_2	x_3	x_5	$\tilde{\phi}_4$
0	0	0	132
1	0	0	144
0	1	0	216
1	1	0	108
0	0	1	132
1	0	1	168
0	1	1	120
1	1	1	60

so that

x_2	x_3	$\sum_{x_5} \tilde{\phi}_4(x_2, x_3, x_5)$		$\tilde{\phi}_{45}$
0	0	132 + 132	=	264
1	0	144 + 168	=	312
0	1	216 + 120	=	336
1	1	108 + 60	=	168

Finally find $\tilde{\phi}_{453}(x_2)$

x_2	x_3	$\phi_A^{x_1=0}$
0	0	4
1	0	2
0	1	2
1	1	6

x_2	x_3	$\tilde{\phi}_{45}$
0	0	264
1	0	312
0	1	336
1	1	168

so that

x_2	$\sum_{x_3} \tilde{\phi}_{45}(x_2, x_3)$		$\tilde{\phi}_{453}$
0	(4 * 264) + (2 * 336)	=	1728
1	(2 * 312) + (6 * 168)	=	1632

The normalising constant is $Z = 1728 + 1632$. Our conditional marginal is thus:

$$p(x_2 \mid x_1 = 0, x_6 = 1) = \begin{pmatrix} 1728/Z \\ 1632/Z \end{pmatrix} = \begin{pmatrix} 0.514 \\ 0.486 \end{pmatrix} \tag{S.1}$$

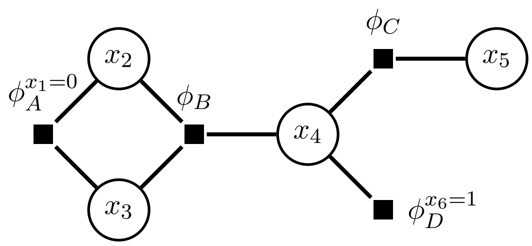
(c) *Note that the previous variable ordering involved computing a new factor $\tilde{\phi}_4$ that depends on three variables x_2, x_3, and x_5, this involved computing 2^3 numbers (i.e. the rows in the table for $\tilde{\phi}_4$). Instead, now find $p(x_2 \mid x_1 = 0, x_6 = 1)$ using the elimination ordering (x_5, x_4, x_3),*

(i) *Draw the graph for $p(x_2, x_3, x_4, \mid x_1 = 0, x_6 = 1)$ by marginalising x_5*
Compute the table for the new factor $\tilde{\phi}_5(x_4)$

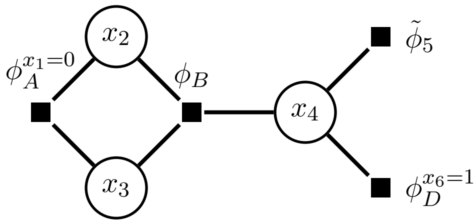
(ii) *Draw the graph for $p(x_2, x_3 \mid x_1 = 0, x_6 = 1)$ by marginalising x_4*
Compute the table for the new factor $\tilde{\phi}_{54}(x_2, x_3)$

(iii) *Draw the graph for $p(x_2 \mid x_1 = 0, x_6 = 1)$ by marginalising x_3*
Compute the table for the new factor $\tilde{\phi}_{543}(x_2)$

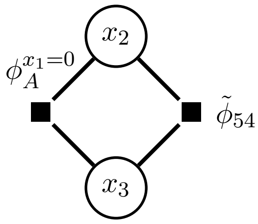
Solution. Starting with the factor graph for $p(x_2, x_3, x_4, x_5 \mid x_1 = 0, x_6 = 1)$



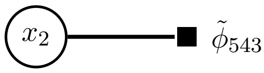
Marginalising x_5 modifies the factor ϕ_C



Marginalising x_4 combines the three factors ϕ_B, $\tilde{\phi}_5$ and $\phi_D^{x_6=1}$



Marginalising x_3 combines the factors $\phi_A^{x_1=0}$ and $\tilde{\phi}_{54}$



We now compute the tables for the new factors $\tilde{\phi}_5$, $\tilde{\phi}_{54}$, and $\tilde{\phi}_{543}$.
First find $\tilde{\phi}_5(x_4)$

x_4	x_5	ϕ_C
0	0	8
1	0	2
0	1	2
1	1	6

so that

x_4	$\sum_{x_5} \phi_C(x_4, x_5)$		$\tilde{\phi}_5$
0	$8 + 2$	$=$	10
1	$2 + 6$	$=$	8

Next find $\tilde{\phi}_{54}(x_2, x_3)$

x_2	x_3	x_4	ϕ_B
0	0	0	2
1	0	0	2
0	1	0	4
1	1	0	2
0	0	1	6
1	0	1	8
0	1	1	4
1	1	1	2

x_4	$\tilde{\phi}_5$
0	10
1	8

x_4	$\phi_D^{x_6=1}$
0	6
1	3

so that

x_2	x_3	$\sum_{x_4} \phi_B(x_2, x_3, x_4)\tilde{\phi}_5(x_4)\phi_D^{x_6=1}(x_4)$		$\tilde{\phi}_{54}$
0	0	$(2 * 10 * 6) + (6 * 8 * 3)$	$=$	264
1	0	$(2 * 10 * 6) + (8 * 8 * 3)$	$=$	312
0	1	$(4 * 10 * 6) + (4 * 8 * 3)$	$=$	336
1	1	$(2 * 10 * 6) + (2 * 8 * 3)$	$=$	168

Finally find $\tilde{\phi}_{543}(x_2)$

x_2	x_3	$\phi_A^{x_1=0}$
0	0	4
1	0	2
0	1	2
1	1	6

x_2	x_3	$\tilde{\phi}_{54}$
0	0	264
1	0	312
0	1	336
1	1	168

so that

x_2	$\sum_{x_3} \tilde{\phi}_{54}(x_2, x_3)$		$\tilde{\phi}_{543}$
0	$(4 * 264) + (2 * 336)$	=	1728
1	$(2 * 312) + (6 * 168)$	=	1632

As with the less efficient ordering in the previous part, we should come to the same result for our conditional marginal distribution.The normalising constant is $Z = 1728 + 1632$, so that the conditional marginal is

$$p(x_2 \mid x_1 = 0, x_6 = 1) = \begin{pmatrix} 1728/Z \\ 1632/Z \end{pmatrix} = \begin{pmatrix} 0.514 \\ 0.486 \end{pmatrix} \tag{S.2}$$

Note that with the first variable ordering (x_4, x_5, x_3) we had to compute 14 numbers $(2^3 + 2^2 + 2^1 = 14)$, but with the better variable ordering (x_5, x_4, x_3) we only needed to compute 8 numbers $(2^1 + 2^2 + 2^1 = 8)$. Choosing a reasonable variable ordering has a direct effect on the computational complexity of variable elimination. This effect becomes even more pronounced when the domain of our discrete variables has a size greater than 2 (binary variables), or if the variables are continuous.