THE UNIVERSITY of EDINBURGH **informatics**

Probabilistic Modelling and Reasoning
Solutions for Tutorial 3

Spring 2020
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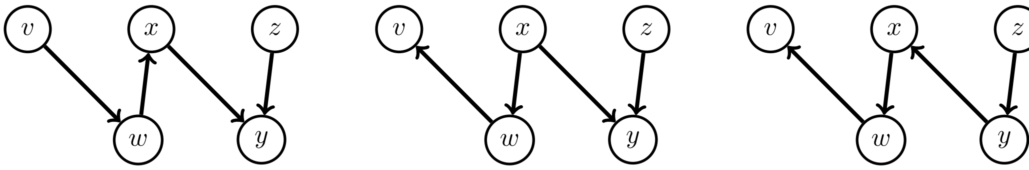
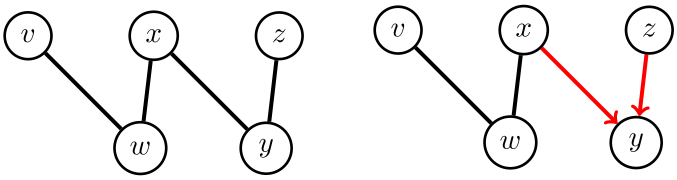
Exercise 1. *I-maps*

(a) *Which of three graphs represent the same set of independencies? Explain.*



Graph 1 *Graph 2* *Graph 3*

Solution. To check whether the graphs are I-equivalent, we have to check the skeletons and the immoralities. All have the same skeleton, but graph 1 and graph 2 also have the same immorality. The answer is thus: graph 1 and 2 encode the same independencies.



skeleton immorality

(b) *Assume that the graph G in Figure 1 is a perfect I-map for $p(a, z, q, e, h)$. Determine the minimal directed I-map using the ordering (e, h, q, z, a). Is the obtained graph I-equivalent to G?*

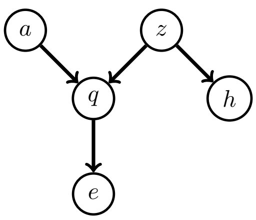


Figure 1: Perfect I-map G for Exercise 1, question (b).

Solution. To find a minimal I-map, we can use the same procedure we used to simplify the factorisation obtained by the chain rule. And since we are given a perfect I-map G for p, we can use the graph to check whether p satisfies a certain independency:

1. Assume an ordering of the variables. Denote the ordered random variables by $x_1, \ldots, x_d$.
2. For each i, find a minimal subset of variables $\pi_i \subseteq \text{pre}_i$ such that

$$x_i \perp\!\!\!\perp \{\text{pre}_i \setminus \pi_i\} \mid \pi_i$$

   is in $\mathcal{I}(G)$ (only works if G is a perfect I-map for $\mathcal{I}(p)$)

3. Construct a graph with parents $\mathrm{pa}_i = \pi_i$.

Note: For I-maps G that are not perfect, if the graph does not indicate that a certain independency holds, we have to check whether the independency indeed does not hold for p. If we don't, we won't obtain a minimal I-map but just an I-map for $\mathcal{I}(p)$. This is because p may have independencies that are not encoded in the graph G. To avoid this difficulty, we here assumed that the graph in Figure 1 is a perfect map; perfect maps represent all independencies, and the lack of an independence assertion by the graph (d-connection) means that the variables are dependent.

Given the ordering (e, h, q, z, a), we build a graph where e is the root. From Figure 1 (and the perfect map assumption), we see that $h \perp\!\!\!\perp e$ does not hold. We thus set e as parent of h, see first graph in Figure 2. Then:

- We consider q: $\mathrm{pre}_q = \{e, h\}$. There is no subset π_q of pre_q on which we could condition to make q independent of $\mathrm{pre}_q \setminus \pi_q$, so that we set the parents of q in the graph to $\mathrm{pa}_q = \{e, h\}$. (Second graph in Figure 2.)
- We consider z: $\mathrm{pre}_z = \{e, h, q\}$. From the graph in Figure 1, we see that for $\pi_z = \{q, h\}$ we have $z \perp\!\!\!\perp \mathrm{pre}_z \setminus \pi_z | \pi_z$. Note that $\pi_z = \{q\}$ does not work because $z \perp\!\!\!\perp e, h | q$ does not hold. We thus set $\mathrm{pa}_z = \{q, h\}$. (Third graph in Figure 2.)
- We consider a: $\mathrm{pre}_a = \{e, h, q, z\}$. This is the last node in the ordering. To find the minimal set π_a for which $a \perp\!\!\!\perp \mathrm{pre}_a \setminus \pi_a | \pi_a$, we can determine its Markov blanket $\mathrm{MB}(a)$. The Markov blanket is the set of parents (none), children (q), and co-parents of a (z) in Figure 1, so that $\mathrm{MB}(a) = \{q, z\}$. We thus set $\mathrm{pa}_a = \{q, z\}$.(Fourth graph in Figure 2.)

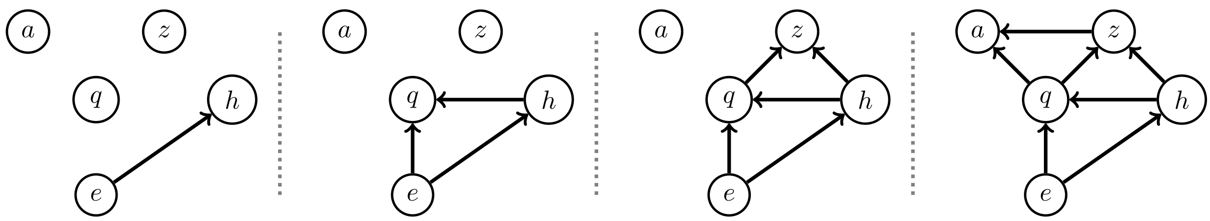


Figure 2: Exercise 1, Question (b):Construction of a minimal directed I-map for the ordering (e, h, q, z, a).

Since the skeleton in the obtained minimal I-map is different from the skeleton of G, we do not have I-equivalence. Note that the ordering (e, h, q, z, a) yields a denser graph (Figure 2) than the graph in Figure 1. While a minimal I-map, the graph does e.g. not show that $a \perp\!\!\!\perp z$. Furthermore, the causal interpretation of the two graphs is different.

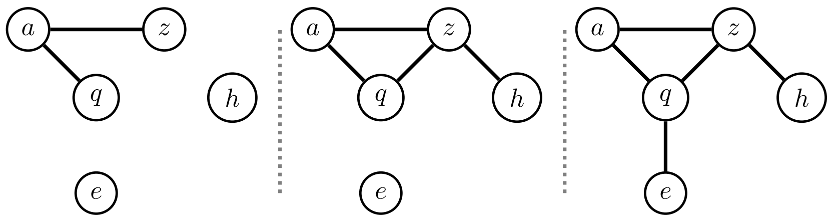
(c) *For the collection of random variables (a, z, h, q, e) you are given the following Markov blankets for each variable:*

- *MB(a) = {q,z}*
- *MB(z) = {a,q,h}*
- *MB(h) = {z}*
- *MB(q) = {a,z,e}*
- *MB(e) = {q}*

(i) *Draw the undirected minimal I-map.*

(ii) *Indicate a Gibbs distribution that satisfies the independence relations specified by the Markov blankets.*

Solution. Connecting each variable to all variables in its Markov blanket yields the desired undirected minimal I-map (see lecture slides). Note that the Markov blankets are not mutually disjoint.



After MB(a) After MB(z) After MB(q)

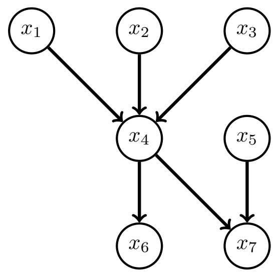
For positive distributions, the set of distributions that satisfy the local Markov property relative to a graph (as given by the Markov blankets) is the same as the set of Gibbs distributions that factorise according to the graph. Given the I-map, we can now easily find the Gibbs distribution

$$p(a, z, h, q, e) = \phi_1(a, z, q)\phi_2(q, e)\phi_3(z, h),$$

ϕ_i must take positive values on their domain. Note that we used the maximal clique (a, z, q).

Exercise 2. *Conversion between graphs*

(a) For the DAG G below find the minimal undirected I-map for $\mathcal{I}(G)$.

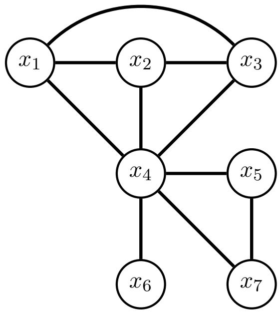


Solution. To derive an undirected minimal I-map from a directed one, we have to construct the moralised graph where the "unmarried" parents are connected by a covering edge. This is because each conditional $p(x_i|\text{pa}_i)$ corresponds to a factor $\phi_i(x_i, \text{pa}_i)$ and we need to connect all variables that are arguments of the same factor with edges.

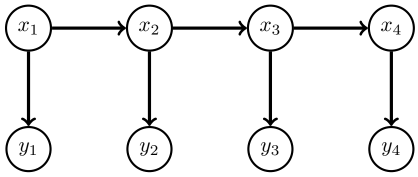
Statistically, the reason for marrying the parents is as follows: An independency $x \perp\!\!\!\perp y|\{\text{child, other nodes}\}$ does not hold in the directed graph in case of collider connections but would hold in the undirected graph if we didn't marry the parents. Hence links between the parents must be added.

It is important to add edges between *all* parents of a node. Here, $p(x_4|x_1, x_2, x_3)$ corresponds to a factor $\phi(x_4, x_1, x_2, x_3)$ so that all four variables need to be connected. Just adding edges $x_1 - x_2$ and $x_2 - x_3$ is not enough.

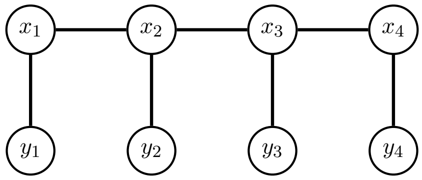
The moral graph, which is the requested minimal undirected I-map, is shown below.



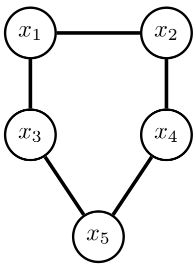
(b) *The DAG below is used to define the hidden Markov model. Determine the undirected minimal I-map for the independencies represented by the DAG.*



Solution. The graph does not contain any head-head connections. The undirected minimal I-map is thus obtained by removing all arrows from the graph.



(c) *Let $\mathcal{U}$ be the independencies that hold for all distributions that factorise over the graph below. Determine the directed minimal I-map for $\mathcal{U}$ with the variable ordering x_1, x_2, x_3, x_4, x_5.*



Solution. We use the ordering x_1, x_2, x_3, x_4, x_5 and follow the general procedure to construct the directed minimal I-map while using the provided undirected graph to check whether an independency is in $\mathcal{U}$ (we can do that because the undirected graph is a perfect I-map for $\mathcal{U}$):

- x_2 is not independent from x_1 so that we set $\text{pa}_2 = \{x_1\}$. See first graph in Figure 3.
- Since x_3 is connected to both x_1 and x_2, we generally don't have $x_3 \perp\!\!\!\perp x_2, x_1$. We cannot make x_3 independent from x_2 by conditioning on x_1 because there are two paths from x_3 to x_2 and x_1 only blocks the upper one. Moreover, x_1 is a neighbour of x_3 so that conditioning on x_2 does make them independent. Hence we must set $\text{pa}_3 = \{x_1, x_2\}$. See second graph in Figure 3.

- For x_4, we see from the undirected graph, that $x_4 \perp\!\!\!\perp x_1 \mid x_3, x_2$. The graph further shows that removing either x_3 or x_2 from the conditioning set is not possible and conditioning on x_1 won't make x_4 independent from x_2 or x_3. We thus have $\mathrm{pa}_4 = \{x_2, x_3\}$. See fourth graph in Figure 3.
- The same reasoning shows that $\mathrm{pa}_5 = \{x_3, x_4\}$. See last graph in Figure 3.

This results in the triangulated directed graph in Figure 3 on the right.

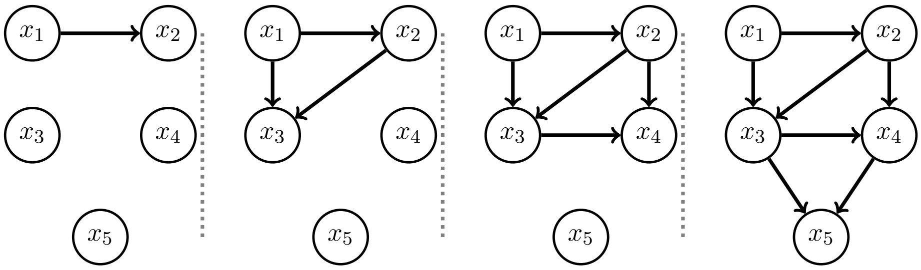


Figure 3: . Answer to Exercise 2, Question (c).

To see why triangulation is necessary consider the case where we didn't have the edge between x_2 and x_3 as in Figure 4. The directed graph would then imply that $x_3 \perp\!\!\!\perp x_2 \mid x_1$ (check!). But this independency assertion does not hold in the undirected graph so that the graph in Figure 4 is not an I-map.

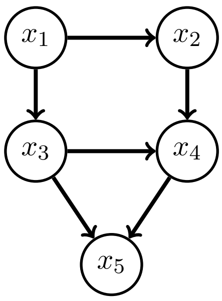
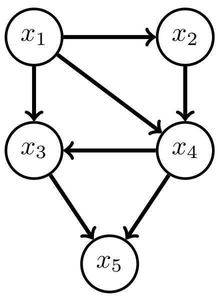


Figure 4: Not a directed I-map for the undirected graphical model defined by the graph in Question(c) of Exercise 2.

(d) *For the undirected graph from question (c) above, which variable ordering yields the directed minimal I-map below?*



Solution. x_1 is the root of the DAG, so it comes first. Next in the ordering are the children of x_1: x_2, x_3, x_4. Since x_3 is a child of x_4, and x_4 a child of x_2, we must have

x_1, x_2, x_4, x_3. Furthermore, x_3 must come before x_5 in the ordering since x_5 is a child of x_3, hence the ordering used must have been: x_1, x_2, x_4, x_3, x_5.

Exercise 3. ***Computer exercise***

Distributed separately. Please check the course homepage.

Solution. *A Python implementation is available on the course homepage.*