Probabilistic Modelling and Reasoning
Tutorial 3 — Additional Practice

Spring 2020
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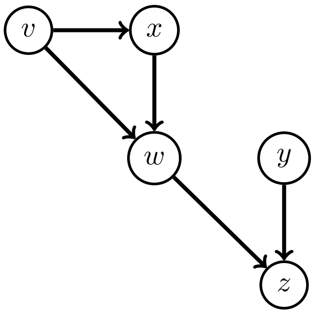
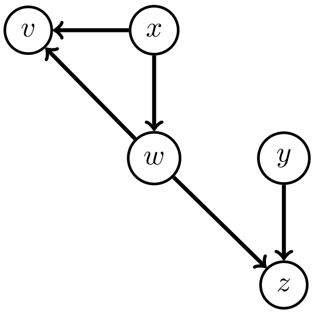
The purpose of this additional sheet is to provide more practice and exam preparation material. N.B. The tutors are not required to work through this material in the tutorial.

Exercise 1. *I-maps*

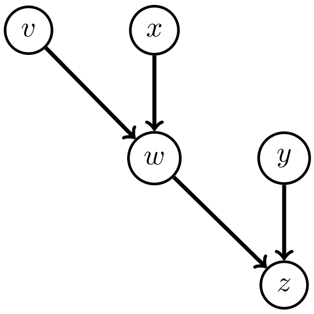
(a) Which of three graphs represent the same set of independencies? Explain.



Graph 1

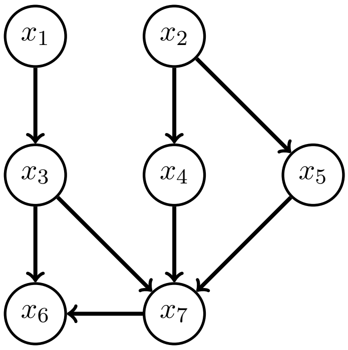


Graph 2



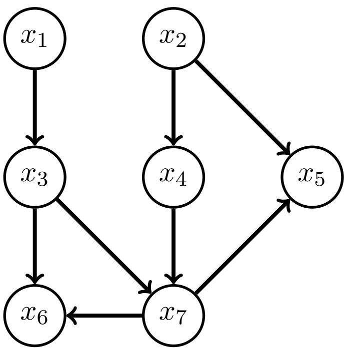
Graph 3

(b) Assume the graph below is a perfect map for a set of independencies $\mathcal{U}$.

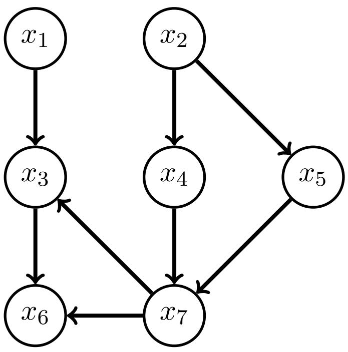


Graph 0

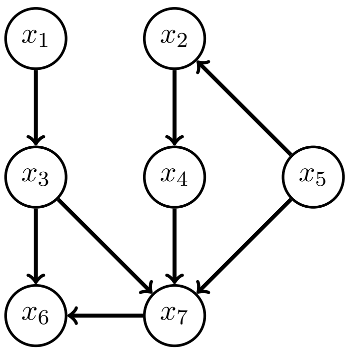
For each of the three graphs, explain whether the graph is a perfect map, an I-map, or not an I-map for $\mathcal{U}$.



Graph 1



Graph 2



Graph 3

Exercise 2. ***Limits of directed and undirected graphical models***

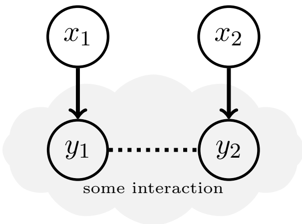
We here consider the probabilistic model $p(y_1, y_2, x_1, x_2) = p(y_1, y_2|x_1, x_2)p(x_1)p(x_2)$ where $p(y_1, y_2|x_1, x_2)$ factorises as

$$p(y_1, y_2|x_1, x_2) = p(y_1|x_1)p(y_2|x_2)\phi(y_1, y_2)n(x_1, x_2) \tag{1}$$

with $n(x_1, x_2)$ equal to

$$n(x_1, x_2) = \left(\int p(y_1|x_1)p(y_2|x_2)\phi(y_1, y_2)\mathrm{d}y_1\mathrm{d}y_2 \right)^{-1}. \tag{2}$$

In the lecture "Factor Graphs", we used the model to illustrate the setup where x_1 and x_2 are two independent inputs that each control the interacting variables y_1 and y_2 (see graph below).



(a) Use the basic characterisations of statistical independence

$$u \perp\!\!\!\perp v|z \Longleftrightarrow p(u, v|z) = p(u|z)p(v|z) \tag{3}$$
$$u \perp\!\!\!\perp v|z \Longleftrightarrow p(u, v|z) = a(u, z)b(v, z) \qquad (a(u, z) \geq 0, b(v, z) \geq 0) \tag{4}$$

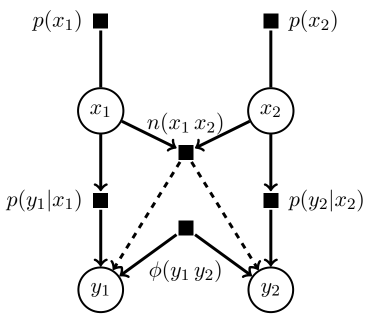
to show that $p(y_1, y_2, x_1, x_2)$ satisfies the following independencies

$$x_1 \perp\!\!\!\perp x_2 \qquad\qquad x_1 \perp\!\!\!\perp y_2 \mid y_1, x_2 \qquad\qquad x_2 \perp\!\!\!\perp y_1 \mid y_2, x_1$$

(b) Is there an undirected perfect map for the independencies satisifed by $p(y_1, y_2, x_1, x_2)$?

(c) Is there a directed perfect map for the independencies satisifed by $p(y_1, y_2, x_1, x_2)$?

(d) *(optional, not examinable)* In the lecture, we have the following factor graph for $p(y_1, y_2, x_1, x_2)$



Use the separation rules for factor graphs to verify that we can find all independence relations. The separation rules are (see Barber, section 4.4.1, or the original paper by Brendan Frey: https://arxiv.org/abs/1212.2486):

"If all paths are blocked, the variables are conditionally independent. A path is blocked if one or more of the following conditions is satisfied:

1. One of the variables in the path is in the conditioning set.
2. One of the variables or factors in the path has two incoming edges that are part of the path (variable or factor collider), and neither the variable or factor nor any of its descendants are in the conditioning set."

Remarks:

- "one or more of the following" should best be read as "one of the following".
- "incoming edges" means directed incoming edges
- the descendants of a variable of factor node are all the variables that you can reach by following a path (containing directed or directed edges, but for directed edges, all directions have to be consistent)
- In the graph we have dashed directed edges: they do count when you determine the descendants but they do not contribute to paths. For example, y_1 is a descendant of the $n(x_1, x_2)$ factor node but $x_1 - n - y_2$ is not a path.