Probabilistic Modelling and Reasoning
Solutions for Tutorial 3 — Additional Practice

Spring 2020
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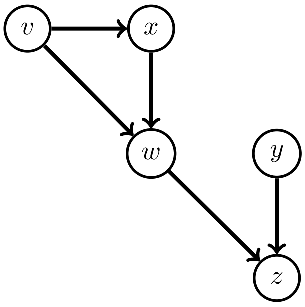
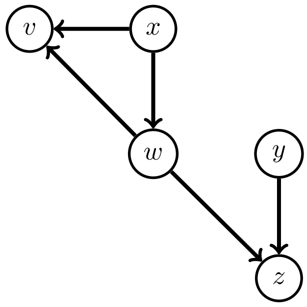
Exercise 1. *I-maps*

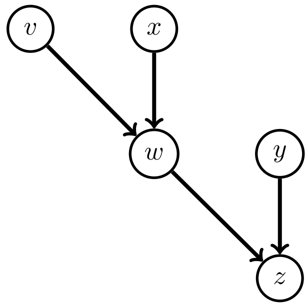
(a) *Which of three graphs represent the same set of independencies? Explain.*



Graph 1

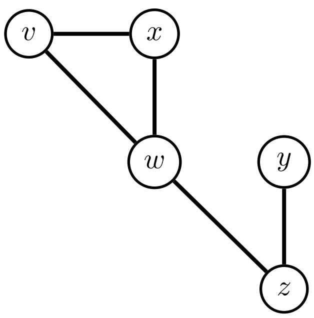


Graph 2

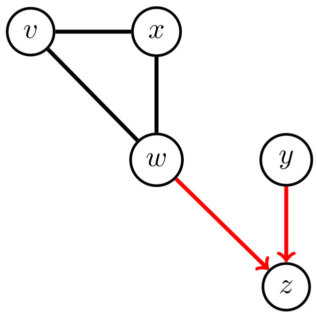


Graph 3

Solution. The skeleton of graph 3 is different from the skeleton of graphs 1 and 2, so that graph 3 cannot be I-equivalent to graph 1 or 2, and we do not need to further check the immoralities for graph 3. Graph 1 and 2 have the same skeleton, and they also have the same immorality. Hence, graph 1 and 2 are I-equivalent. Note that node w in graph 1 is in a collider configuration along trail $v - w - x$ but it is not an immorality because its parents are connected (covering edge); equivalently for node v in graph 2.

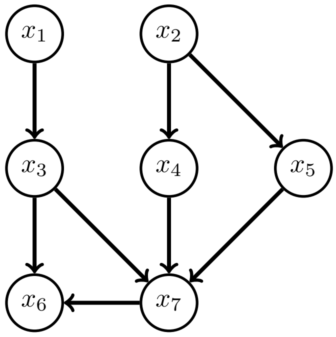


skeleton



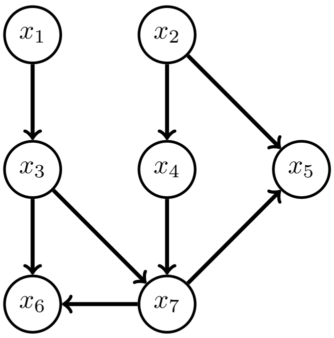
immorality

(b) *Assume the graph below is a perfect map for a set of independencies* $\mathcal{U}$.

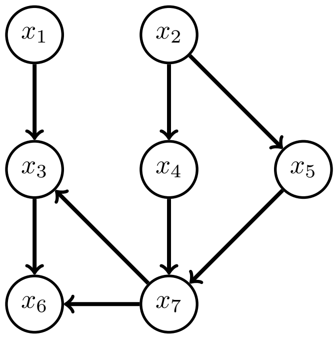


Graph 0

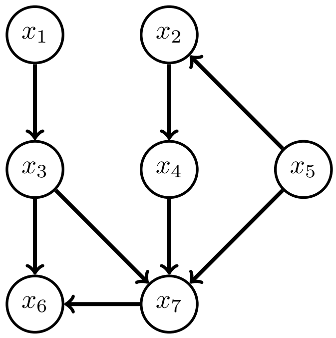
For each of the three graphs, explain whether the graph is a perfect map, an I-map, or not an I-map for $\mathcal{U}$.



Graph 1



Graph 2



Graph 3

Solution.

- Graph 1 has an immorality $x_2 \to x_5 \leftarrow x_7$ which graph 0 does not have. The graph is thus not I-equivalent to graph 0 and can thus not be a perfect map. Moreover, graph 1 asserts that $x_2 \perp\!\!\!\perp x_7 | x_4$ which is not case for graph 0. Since graph 0 is a perfect map for $\mathcal{U}$, graph 1 asserts an independency that does not hold for $\mathcal{U}$ and can thus not be an I-map for $\mathcal{U}$.
- Graph 2 has an immorality $x_1 \to x_3 \leftarrow x_7$ which graph 0 does not have. Graph 2 thus asserts that $x_1 \perp\!\!\!\perp x_7$, which is not the case for graph 0. Hence, for the same reason as for graph 1, graph 2 is not an I-map for $\mathcal{U}$.
- Graph 3 has the same skeleton and set of immoralities as graph 0. It is thus I-equivalent to graph 0, and hence also a perfect map.

Exercise 2. ***Limits of directed and undirected graphical models***

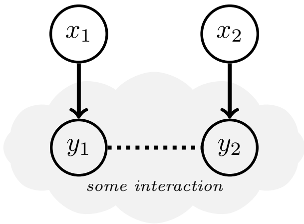
We here consider the probabilistic model $p(y_1, y_2, x_1, x_2) = p(y_1, y_2|x_1, x_2)p(x_1)p(x_2)$ where $p(y_1, y_2|x_1, x_2)$ factorises as

$$p(y_1, y_2|x_1, x_2) = p(y_1|x_1)p(y_2|x_2)\phi(y_1, y_2)n(x_1, x_2) \tag{1}$$

with $n(x_1, x_2)$ equal to

$$n(x_1, x_2) = \left(\int p(y_1|x_1)p(y_2|x_2)\phi(y_1, y_2)\mathrm{d}y_1\mathrm{d}y_2 \right)^{-1}. \tag{2}$$

In the lecture "Factor Graphs", we used the model to illustrate the setup where x_1 and x_2 are two independent inputs that each control the interacting variables y_1 and y_2 (see graph below).



(a) *Use the basic characterisations of statistical independence*

$$u \perp\!\!\!\perp v|z \iff p(u,v|z) = p(u|z)p(v|z) \tag{3}$$

$$u \perp\!\!\!\perp v|z \iff p(u,v|z) = a(u,z)b(v,z) \qquad (a(u,z) \geq 0, b(v,z) \geq 0) \tag{4}$$

to show that $p(y_1, y_2, x_1, x_2)$ *satisfies the following independencies*

$$x_1 \perp\!\!\!\perp x_2 \qquad\qquad x_1 \perp\!\!\!\perp y_2 \mid y_1, x_2 \qquad\qquad x_2 \perp\!\!\!\perp y_1 \mid y_2, x_1$$

Solution. The pdf/pmf is

$$p(y_1, y_2, x_1, x_2) = p(y_1|x_1)p(y_2|x_2)\phi(y_1, y_2)n(x_1, x_2)p(x_1)p(x_2)$$

For $\mathbf{x_1} \perp\!\!\!\perp \mathbf{x_2}$

We compute $p(x_1, x_2)$ as

$$p(x_1, x_2) = \int p(y_1, y_2, x_1, x_2)\mathrm{d}y_1\mathrm{d}y_2 \tag{S.1}$$

$$= \int p(y_1|x_1)p(y_2|x_2)\phi(y_1, y_2)n(x_1, x_2)p(x_1)p(x_2)\mathrm{d}y_1\mathrm{d}y_2 \tag{S.2}$$

$$= n(x_1, x_2)p(x_1)p(x_2) \int p(y_1|x_1)p(y_2|x_2)\phi(y_1, y_2)\mathrm{d}y_1\mathrm{d}y_2 \tag{S.3}$$

$$\overset{(2)}{=} n(x_1, x_2)p(x_1)p(x_2)\frac{1}{n(x_1, x_2)} \tag{S.4}$$

$$= p(x_1)p(x_2). \tag{S.5}$$

Since $p(x_1)$ and $p(x_2)$ are the univariate marginals of x_1 and x_2, respectively, it follows from (3) that $x_1 \perp\!\!\!\perp x_2$.

For $\mathbf{x_1} \perp\!\!\!\perp \mathbf{y_2} \mid \mathbf{y_1}, \mathbf{x_2}$

We rewrite $p(y_1, y_2, x_1, x_2)$ as

$$p(y_1, y_2, x_1, x_2) = p(y_1|x_1)p(y_2|x_2)\phi(y_1, y_2)n(x_1, x_2)p(x_1)p(x_2) \tag{S.6}$$

$$= [p(y_1|x_1)p(x_1)n(x_1, x_2)]\,[p(y_2|x_2)\phi(y_1, y_2)p(x_2)] \tag{S.7}$$

$$= \phi_A(x_1, y_1, x_2)\phi_B(y_2, y_1, x_2) \tag{S.8}$$

With (4), we have that $x_1 \perp\!\!\!\perp y_2 \mid y_1, x_2$. Note that $p(x_2)$ can be associated either with ϕ_A or with ϕ_B.

For $\mathbf{x_2} \perp\!\!\!\perp \mathbf{y_1} \mid \mathbf{y_2}, \mathbf{x_1}$

We use here the same approach as for $x_1 \perp\!\!\!\perp y_2 \mid y_1, x_2$. (By symmetry considerations, we could immediately see that the relation holds but let us write it out for clarity). We rewrite $p(y_1, y_2, x_1, x_2)$ as

$$p(y_1, y_2, x_1, x_2) = p(y_1|x_1)p(y_2|x_2)\phi(y_1, y_2)n(x_1, x_2)p(x_1)p(x_2) \tag{S.9}$$

$$= [p(y_2|x_2)n(x_1, x_2)p(x_2)p(x_1))]\,[p(y_1|x_1)\phi(y_1, y_2)]) \tag{S.10}$$

$$= \tilde{\phi}_A(x_2, x_1, y_2)\tilde{\phi}_B(y_1, y_2, x_1) \tag{S.11}$$

With (4), we have that $x_2 \perp\!\!\!\perp y_1 \mid y_2, x_1$.

(b) *Is there an undirected perfect map for the independencies satisifed by $p(y_1, y_2, x_1, x_2)$?*

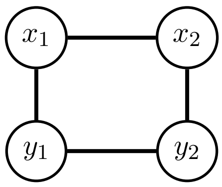
Solution. We write

$$p(y_1, y_2, x_1, x_2) = p(y_1|x_1)p(y_2|x_2)\phi(y_1, y_2)n(x_1, x_2)p(x_1)p(x_2)$$

as a Gibbs distribution

$$p(y_1, y_2, x_1, x_2) = \phi_1(y_1, x_1)\phi_2(y_2, x_2)\phi_3(y_1, y_2)\phi_4(x_1, x_2) \qquad \text{with} \tag{S.12}$$
$$\phi_1(y_1, x_1) = p(y_1|x_1)p(x_1) \tag{S.13}$$
$$\phi_2(y_2, x_2) = p(y_2|x_2)p(x_2) \tag{S.14}$$
$$\phi_3(y_1, y_2) = \phi(y_1, y_2) \tag{S.15}$$
$$\phi_4(x_1, x_2) = n(x_1, x_2). \tag{S.16}$$

Visualising it as an undirected graph gives an I-map:



While the graph implies $x_1 \perp\!\!\!\perp y_2 \mid y_1, x_2$ and $x_2 \perp\!\!\!\perp y_1 \mid y_2, x_1$, the independency $x_1 \perp\!\!\!\perp x_2$ is not represented. Hence the graph is not a perfect map. Note further that removing any edge would result in a graph that is not an I-map for $\mathcal{I}(p)$ anymore. Hence the graph is a minimal I-map for $\mathcal{I}(p)$ but that we cannot obtain a perfect I-map.

(c) *Is there a directed perfect map for the independencies satisifed by $p(y_1, y_2, x_1, x_2)$?*

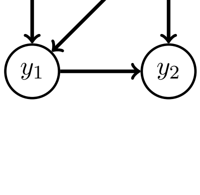
Solution. We construct directed minimal I-maps for $p(y_1, y_2, x_1, x_2) = p(y_1, y_2|x_1, x_2)p(x_1)p(x_2)$ for different orderings as explained in the lecture. We will see that they do not represent all independencies in $\mathcal{I}(p)$ and hence that they are not perfect I-maps.

To guarantee unconditional independence of x_1 and x_2, the two variables must come first in the orderings (either x_1 and then x_2 or the other way around).

If we use the ordering x_1, x_2, y_1, y_2, and that

- $x_1 \perp\!\!\!\perp x_2$
- $y_2 \perp\!\!\!\perp x_1 | y_1, x_2$, which is $y_2 \perp\!\!\!\perp \text{pre}(y_2) \setminus \pi | \pi$ for $\pi = (y_1, x_2)$

are in $\mathcal{I}(p)$, we obtain the following directed minimal I-map:



The graphs misses $x_2 \perp\!\!\!\perp y_1 \mid y_2, x_1$.

If we use the ordering x_1, x_2, y_2, y_1, and that

- $x_1 \perp\!\!\!\perp x_2$
- $y_1 \perp\!\!\!\perp x_2 | x_1, y_2$, which is $y_1 \perp\!\!\!\perp \text{pre}(y_1) \setminus \pi | \pi$ for $\pi = (x_1, y_2)$

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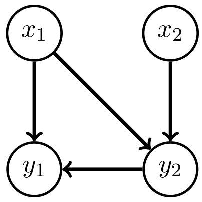
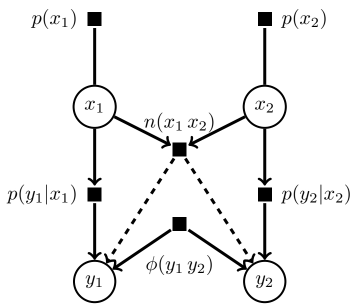


The graph misses $x_1 \perp\!\!\!\perp y_2 \mid y_1, x_2$.

Moreover, the graphs imply a directionality between y_1 and y_2, or a direct influence of x_1 on y_2, or of x_2 on y_1, in contrast to the original modelling goals.

(d) (optional, not examinable) *In the lecture, we have the following factor graph for* $p(y_1, y_2, x_1, x_2)$



Use the separation rules for factor graphs to verify that we can find all independence relations. The separation rules are (see Barber, section 4.4.1, or the original paper by Brendan Frey: https://arxiv.org/abs/1212.2486):

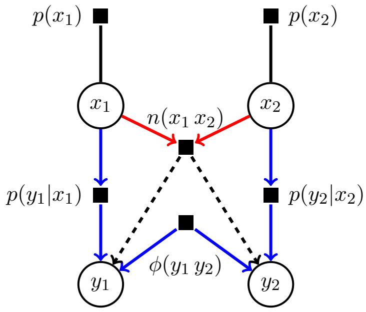
"If all paths are blocked, the variables are conditionally independent. A path is blocked if one or more of the following conditions is satisfied:

1. *One of the variables in the path is in the conditioning set.*
2. *One of the variables or factors in the path has two incoming edges that are part of the path (variable or factor collider), and neither the variable or factor nor any of its descendants are in the conditioning set."*

Remarks:

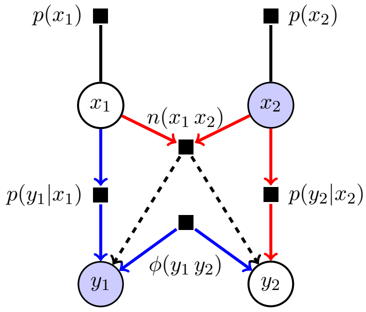
- *"one or more of the following" should best be read as "one of the following".*
- *"incoming edges" means directed incoming edges*
- *the descendants of a variable of factor node are all the variables that you can reach by following a path (containing directed or directed edges, but for directed edges, all directions have to be consistent)*
- *In the graph we have dashed directed edges: they do count when you determine the descendants but they do not contribute to paths. For example, y_1 is a descendant of the $n(x_1, x_2)$ factor node but $x_1 - n - y_2$ is not a path.*

Solution. $\mathbf{x_1} \perp\!\!\!\perp \mathbf{x_2}$
There are two paths from x_1 to x_2 marked with red and blue below:



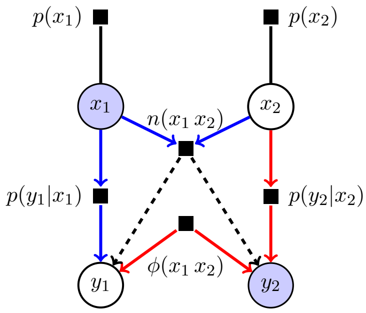
Both the blue and red path are blocked by condition 2.

$\mathbf{x_1} \perp\!\!\!\perp \mathbf{y_2} \mid \mathbf{y_1}, \mathbf{x_2}$
There are two paths from x_1 to y_2 marked with red and blue below:



The observed variables are marked in blue. For the red path, the observed x_2 blocks the path (condition 1). Note that the $n(x_1, x_2)$ node would be open by condition 2. The blue path is blocked by condition 1 too. In directed graphical models, the y_1 node would be open, but here while condition 2 does not apply, condition 1 still applies (note the *one or more of ...* in the separation rules), so that the path is blocked.

$\mathbf{x_2} \perp\!\!\!\perp \mathbf{y_1} \mid \mathbf{y_2}, \mathbf{x_1}$
There are two paths from x_2 to y_1 marked with red and blue below:



The same reasoning as before yields the result.

Finally note that x_1 and x_2 are not independent given y_1 or y_2 because the upper path through $n(x_1, x_2)$ is not blocked whenever y_1 or y_2 are observed (condition 2).

Credit: this example is discussed in the original paper by B. Frey (Figure 6).