Factor Graphs

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Recap

- Undirected and directed graphical models have complementary properties
- Both encode and (visually) represent statistical independencies (I-maps)
- Graphs tell us how probability density/mass functions factorise
- For directed graphs with parent sets pa_i

$$p(x_1, \ldots, x_d) = \prod_{i=1}^{d} p(x_i | \mathrm{pa}_i)$$

- For undirected graphs with maximal clique sets $\mathcal{X}_c$

$$p(x_1, \ldots, x_d) = \frac{1}{Z} \prod_c \phi_c(\mathcal{X}_c)$$

Program

1. What are factor graphs?
2. Advantages over directed or undirected graphs?

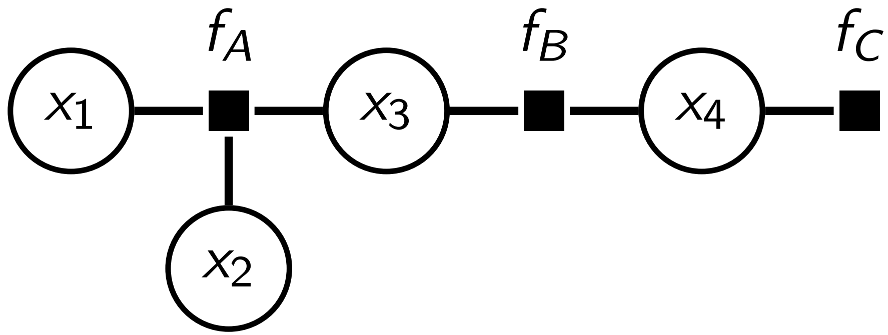
Program

1. What are factor graphs?
   - Definition
   - Visualising Gibbs distributions as factor graphs
   - Visualising factors that are conditionals
2. Advantages over directed or undirected graphs?

Definition of factor graphs

- A factor graph represents the factorisation of an arbitrary function (not necessarily related to pdfs/pmfs)
- Example: $h(x_1, x_2, x_3, x_4) = f_A(x_1, x_2, x_3) f_B(x_3, x_4) f_C(x_4)$

  Factor graph (FG):



- Two types of nodes: factor and variable nodes
- Convention: squares for factors, circles for variables (other conventions are used too)

Definition of factor graphs

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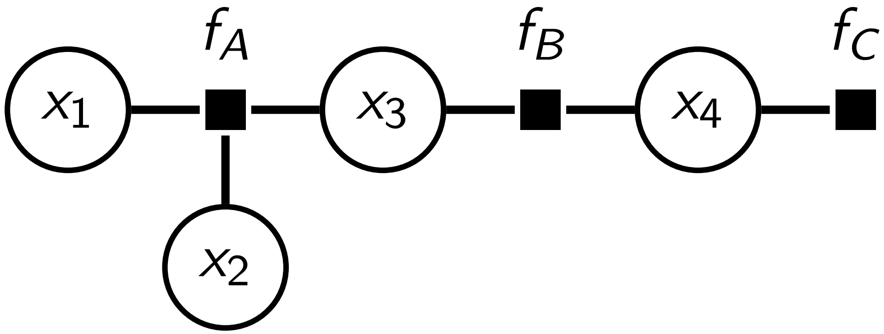
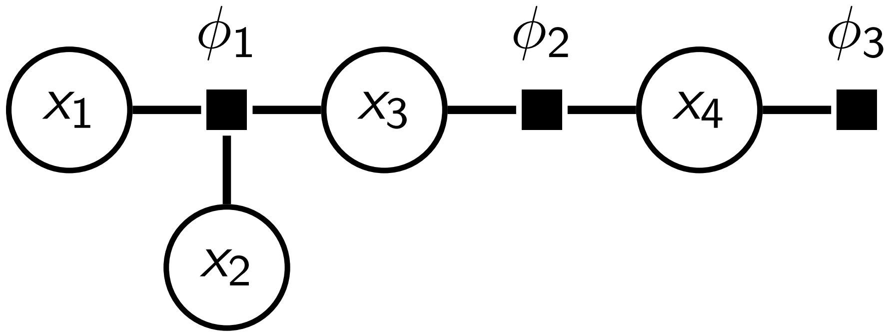
  Factor graph (FG):



- Edge between variable x and factor f $\Leftrightarrow$ x is an argument of f
- Variable nodes are always connected to factor nodes; no direct links between factor or variable nodes (FGs are bipartite graphs)
- We can also use directed edges (to indicate conditionals)

Visualising Gibbs distributions as factor graphs

- Example: $p(x_1, x_2, x_3, x_4) = \frac{1}{Z}\phi_1(x_1, x_2, x_3)\phi_2(x_3, x_4)\phi_3(x_4)$



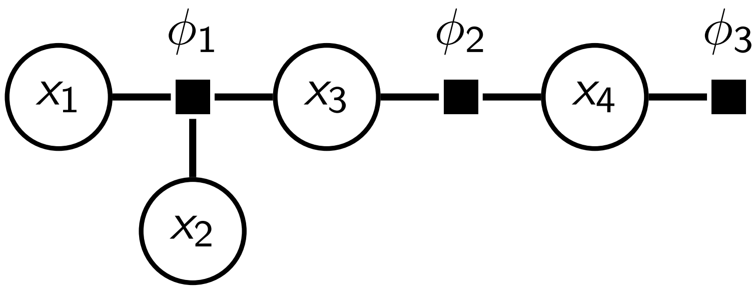
- General case: $p(x_1, \ldots, x_d) \propto \prod_c \phi_c(\mathcal{X}_c)$
  - Factor node for all ϕ_c
  - For all factors ϕ_c:
    draw an undirected edge between ϕ_c and all $x_i \in \mathcal{X}_c$.
- Can visualise any undirected graphical model as a factor graph.

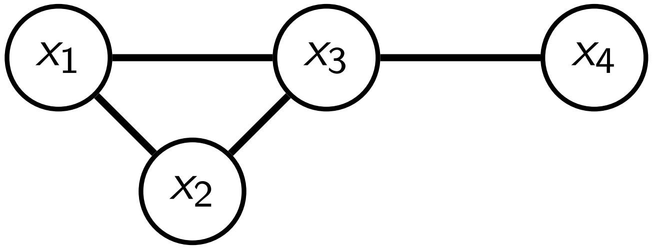
Visualising Gibbs distributions as factor graphs

Some differences to visualisation with undirected graph:

- Factors ϕ_c are shown; makes the graphs more informative (see next slide)
- Variables x_i are neighbours if they are connected to the same factor.

$p(x_1, x_2, x_3, x_4) = \frac{1}{Z}\phi_1(x_1, x_2, x_3)\phi_2(x_3, x_4)\phi_3(x_4)$

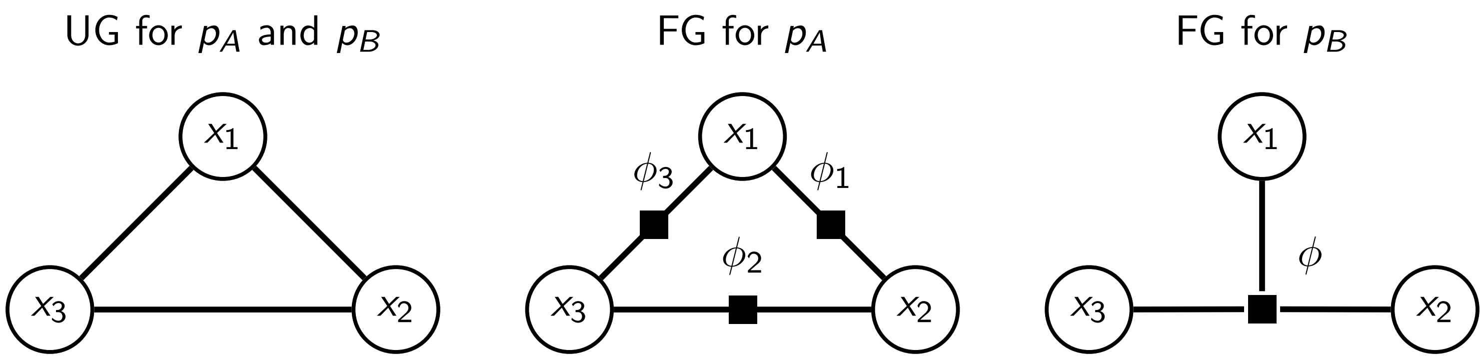




More informative than undirected graphs

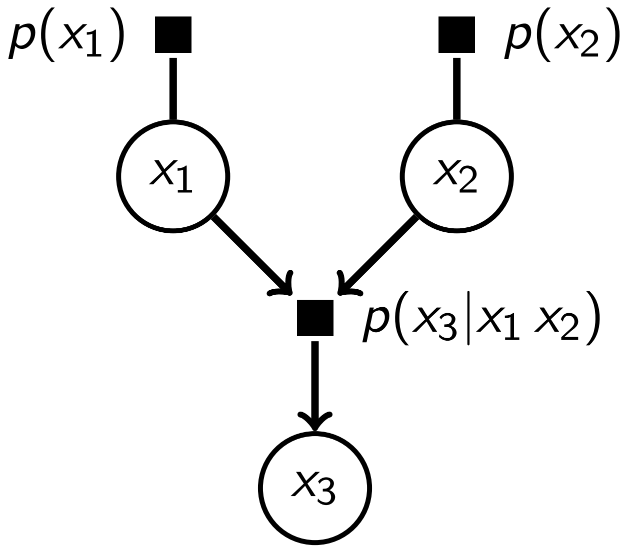
- Mapping from Gibbs distribution to undirected graph is many to one but one-to-one for factor graphs.
- Example

$$p_A(x_1, x_2, x_3) \propto \phi_1(x_1, x_2)\phi_2(x_2, x_3)\phi_3(x_3, x_1)$$
$$p_B(x_1, x_2, x_3) \propto \phi(x_1, x_2, x_3)$$



Visualising factors that are conditionals

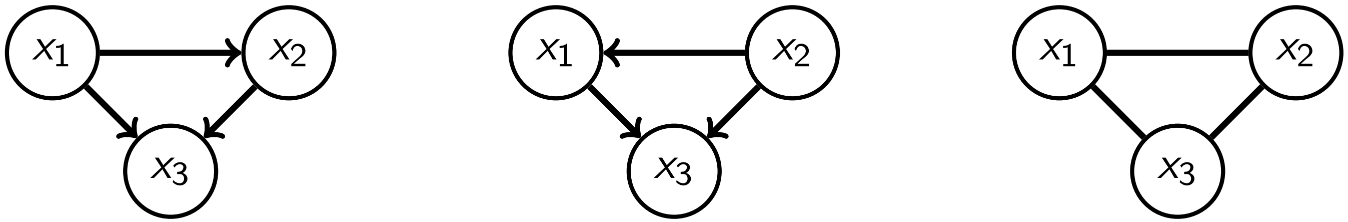
- For $p(x_1, x_2, x_3) = p(x_1)p(x_2)p(x_3|x_1, x_2)$, we may want to include the information that x_3 is conditioned on x_1, x_2
- Use arrows as in directed graphs.



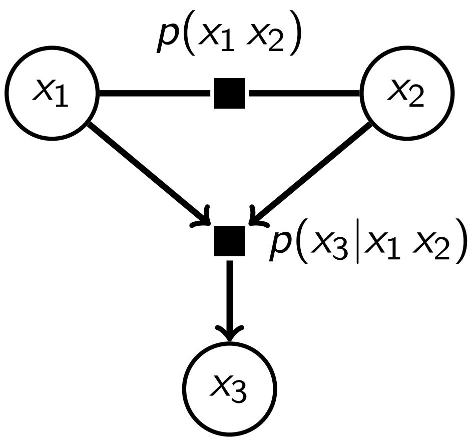
- Can visualise any directed graphical model as a factor graph.

Mixed graphs

- Let $p(x_1, x_2, x_3) = p(x_1, x_2)p(x_3|x_1, x_2)$.
- Directed graphs forces ordering of the random variables; undirected graph does not show conditioning on x_1, x_2



- Mixed FG to visualise the conditioning for $p(x_3|x_1, x_2)$ without imposing an ordering on x_1 and x_2



Program

1. What are factor graphs?

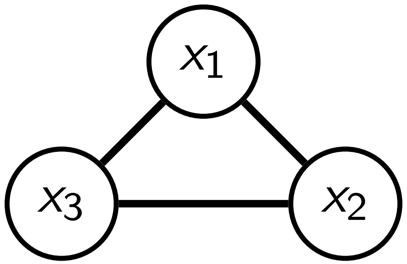
2. Advantages over directed or undirected graphs?
   - Computational advantages
   - Statistical advantages

Importance of factorisation

- Factorisation was central in the development so far
- But directed and undirected graphs are not able to fully represent arbitrary factorisations of pdfs/pmfs.

  For example, same graph for

$$p(x_1, x_2, x_3) \propto \phi_1(x_1, x_2)\phi_2(x_2, x_3)\phi_3(x_3, x_1)$$
$$p(x_1, x_2, x_3) \propto \phi(x_1, x_2, x_3)$$

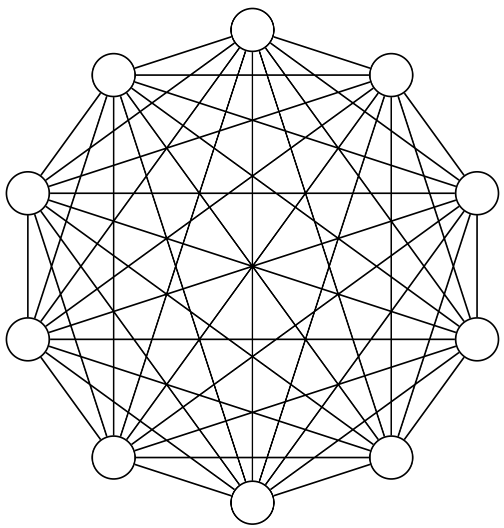


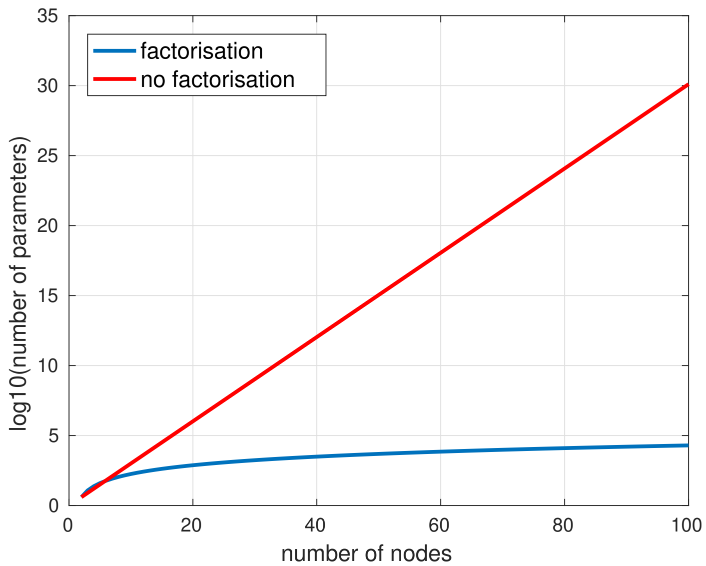
- We should expect that being able to better represent the factorisation has advantages.

Example of computational advantages

Assume binary random variables x_i.

- Same undirected graph but
  $p(x_1, \ldots, x_d) \propto \phi(x_1, \ldots, x_d)$ has 2^d free parameters,
  $p(x_1, \ldots, x_d) \propto \prod_{i<j} \phi_{ij}(x_i, x_j)$ has $\binom{d}{2}2^2$ free parameters
  parameters $\equiv$ entries to specify in a table representation
- The difference matters for learning and inference when the number of variables is large.

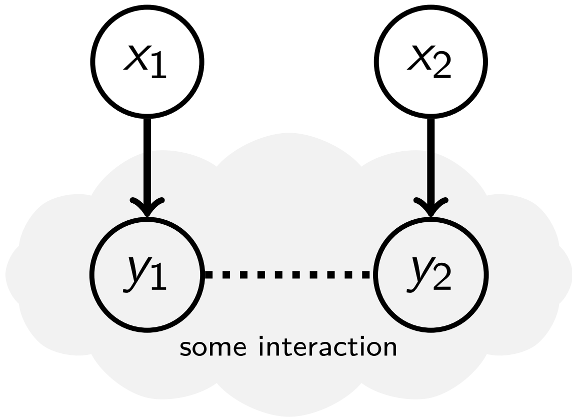




Example of statistical advantages

- Let x_1 and x_2 be two inputs
- x_1 controls variable y_1

  x_2 controls y_2
- Variables y_1 and y_2 influence each other



- Model: $p(y_1, y_2, x_1, x_2) = p(y_1, y_2 | x_1, x_2) p(x_1) p(x_2)$

  (probabilistic modelling: pdf/pmf $p(y_1, y_2 | x_1, x_2)$ captures uncertainty about how the x_i affect the y_i and about how the y_i interact)

- Choose $p(y_1, y_2 | x_1, x_2)$ such that $p(y_1, y_2, x_1, x_2)$ satisfies
  - $x_1 \perp\!\!\!\perp x_2$ (independence between control variables)
  - $x_1 \perp\!\!\!\perp y_2 \mid y_1, x_2$ (y_2 is only directly influenced by y_1 and x_2)
  - $x_2 \perp\!\!\!\perp y_1 \mid y_2, x_1$ (y_1 is only directly influenced by y_2 and x_1)

Example of statistical advantages

- Three independencies are satisfied if $p(y_1, y_2|x_1, x_2)$ factorises as

$$p(y_1, y_2|x_1, x_2) = p(y_1|x_1)p(y_2|x_2)\phi(y_1, y_2)n(x_1, x_2)$$

where $n(x_1, x_2)$ ensures normalisation of $p(y_1, y_2|x_1, x_2)$

$$n(x_1, x_2) = \left(\int p(y_1|x_1)p(y_2|x_2)\phi(y_1, y_2)\mathrm{d}y_1\mathrm{d}y_2\right)^{-1}$$

(see tutorials)

- Directed and undirected graphs cannot represent the independencies induced by factorisation of $p(y_1, y_2|x_1, x_2)$ (see tutorials).
- Factor graphs and chain graphs (see Barber, Section 4.3, not covered in lecture) can represent them.
- Factor graphs can represent independencies that DAGs or UGs cannot or do not represent.

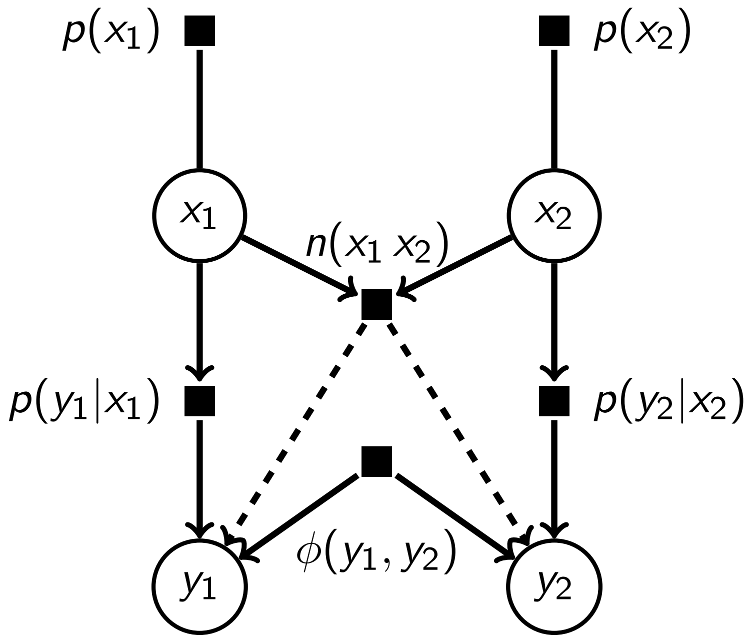
Example of statistical advantages

(not examinable)

- Overall model:

$$p(y_1, y_2, x_1, x_2) = \overbrace{p(y_1|x_1)p(y_2|x_2)\phi(y_1, y_2)n(x_1, x_2)}^{p(y_1,y_2|x_1,x_2)} p(x_1)p(x_2)$$

- Factor graph (Note: directed edges to y_1, y_2 for all factors involved in the conditional)



- Independencies can be found from separation rules for factor graphs (see Barber, Section 4.4.1, and original paper "Extending Factor Graphs so as to Unify Directed and Undirected Graphical Models" by B. Frey, UAI 2003).

Program recap

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   - Statistical advantages