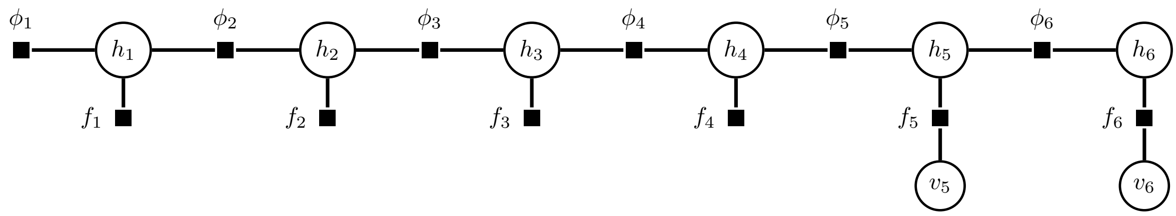
Exercise 1. *Hidden Markov model – beta-recursion*

We consider the following factor graph from the lecture on hidden Markov models.



The factor graph corresponds to the conditional pmf

$$p(h_1, \ldots, h_6, v_5, v_6 \mid v_{1:4})$$

and the factors are defined as

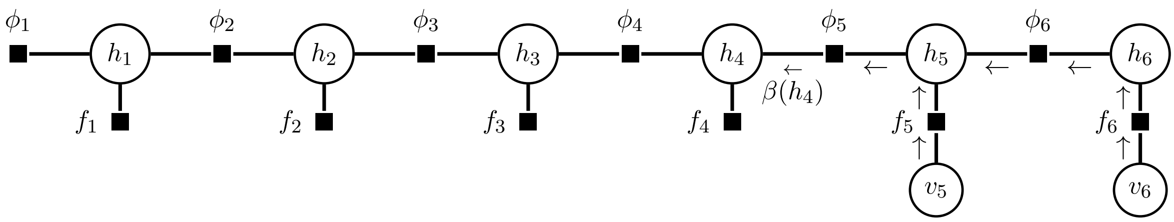
$$f_t(h_t) = p(v_t|h_t) \quad (t \leq 4) \qquad f_t(v_t, h_t) = p(v_t|h_t) \quad (t > 4) \tag{1}$$

$$\phi_1(h_1) = p(h_1) \qquad \phi_t(h_t, h_{t-1}) = p(h_t|h_{t-1}) \quad (t > 1) \tag{2}$$

We define $\beta(h_s) = \mu_{\phi_{s+1} \to h_s}(h_s)$, which is the message from a factor node "back" to a variable node.

(a) *Show that $\beta(h_4) = \mu_{\phi_5 \to h_4}(h_4) = 1$.*

Solution. The arrows in the factor graph below show the messages that need to be computed for the computation of $\beta(h_4)$.



We start with the leaf variable v_6:

$$\mu_{v_6 \to f_6}(v_6) = 1 \tag{S.1}$$

$$\mu_{f_6 \to h_6}(h_6) = \sum_{v_6} f_6(v_6, h_6)\mu_{v_6 \to f_6}(v_6) \tag{S.2}$$

$$= \sum_{v_6} p(v_6|h_6) \cdot 1 \tag{S.3}$$

$$= 1 \quad \text{since (conditional) pmfs and pdfs are normalised} \tag{S.4}$$

The variable node h_6, having a single incoming message only, copies the message so that

$$\mu_{h_6 \to \phi_6}(h_6) = \beta(h_6) = 1. \tag{S.5}$$

For the next message, which corresponds to the elimination of h_6, we have:

$$\begin{aligned}
\mu_{\phi_6 \to h_5}(h_5) &= \sum_{h_6} \phi_6(h_6, h_5)\mu_{h_6 \to \phi_6}(h_6) && \text{(S.6)}\\
&= \sum_{h_6} p(h_6|h_5) \cdot 1 && \text{(S.7)}\\
&= 1 \quad \text{since (conditional) pmfs and pdfs are normalised.} && \text{(S.8)}
\end{aligned}$$

The same kind of calculations show that $\mu_{f_5 \to h_5} = 1$. It follows that

$$\begin{aligned}
\mu_{h_5 \to \phi_5}(x_5) &= \mu_{\phi_6 \to h_5}(h_5)\mu_{f_5 \to h_5} && \text{(S.9)}\\
&= 1. && \text{(S.10)}
\end{aligned}$$

We thus obtain the desired result for $\beta(h_4) = \mu_{\phi_5 \to h_4}(h_4)$:

$$\begin{aligned}
\mu_{\phi_5 \to h_4}(h_4) &= \sum_{h_5} \phi_5(h_5, h_4)\mu_{h_5 \to \phi_5}(x_5) && \text{(S.11)}\\
&= \sum_{x_5} p(h_5|h_4) \cdot 1 && \text{(S.12)}\\
&= 1 \quad \text{since (conditional) pmfs and pdfs are normalised.} && \text{(S.13)}
\end{aligned}$$

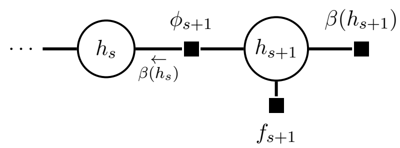
(b) *Use sum-product message passing to show that the beta-recursion holds*

$$\begin{aligned}
\beta(h_4) &= 1 && (3)\\
\beta(h_s) &= \sum_{h_{s+1}} p(h_{s+1}|h_s)p(v_{s+1}|h_{s+1})\beta(h_{s+1}) \quad (s < 4) && (4)
\end{aligned}$$

Solution. We defined $\beta(h_s)$ as the message $\mu_{\phi_{s+1} \to h_s}(h_s)$. We thus also have

$$\beta(h_{s+1}) = \mu_{\phi_{s+2} \to h_{s+1}}(h_{s+1}), \tag{S.14}$$

which is the effective factor for h_{s+1} if all variables in all sub-trees attached to ϕ_{s+2}, with exception of the sub-trees attached to h_{s+1}, are eliminated. This gives us the following fragment of a factor graph



Message passing tell us that

$$\beta(h_s) = \mu_{\phi_{s+1} \to h_s}(h_s) = \sum_{h_{s+1}} \phi_{s+1}(h_{s+1}, h_s)\mu_{h_{s+1} \to \phi_{s+1}}(h_{s+1}) \tag{S.15}$$

and that

$$\begin{aligned}
\mu_{h_{s+1} \to \phi_{s+1}}(h_{s+1}) &= \mu_{f_{s+1} \to h_{s+1}}(h_{s+1})\mu_{\beta(h_{s+1}) \to h_{s+1}}(h_{s+1}) && \text{(S.16)}\\
&= f_{s+1}(h_{s+1})\beta(h_{s+1}), && \text{(S.17)}\\
& && \text{(S.18)}
\end{aligned}$$

where for the last equation, we have used that f_{s+1} and $\beta(h_{s+1})$ are leaf factor nodes. We thus obtain

$$\beta(h_s) = \mu_{\phi_{s+1} \to h_s}(h_s) = \sum_{h_{s+1}} \phi_{s+1}(h_{s+1}, h_s) f_{s+1}(h_{s+1}) \beta(h_{s+1}). \tag{S.19}$$

Plugging in the definition of the factors gives

$$\beta(h_s) = \sum_{h_{s+1}} p(h_{s+1}|h_s) p(v_{s+1}|h_{s+1}) \beta(h_{s+1}), \tag{S.20}$$

which is the desired recursion. In our factor graph, the recursion is initialised with $\beta(h_4) = 1$.