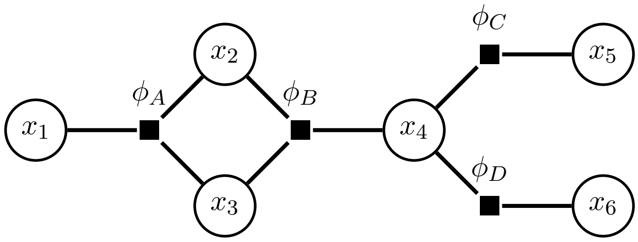
*The purpose of this additional sheet is to provide more practice and exam preparation material. N.B. The tutors are not required to work through this material in the tutorial.*

**Exercise 1.** ***Choice of elimination order in factor graphs***

Consider the following factor graph, which contains a loop:




Let all variables be binary, $x_i \in \{0, 1\}$, and the factors be defined as follows:

| $x_1$ | $x_2$ | $x_3$ | $\phi_A$ |
|---|---|---|---|
| 0 | 0 | 0 | 4 |
| 1 | 0 | 0 | 2 |
| 0 | 1 | 0 | 2 |
| 1 | 1 | 0 | 6 |
| 0 | 0 | 1 | 2 |
| 1 | 0 | 1 | 6 |
| 0 | 1 | 1 | 6 |
| 1 | 1 | 1 | 4 |

| $x_2$ | $x_3$ | $x_4$ | $\phi_B$ |
|---|---|---|---|
| 0 | 0 | 0 | 2 |
| 1 | 0 | 0 | 2 |
| 0 | 1 | 0 | 4 |
| 1 | 1 | 0 | 2 |
| 0 | 0 | 1 | 6 |
| 1 | 0 | 1 | 8 |
| 0 | 1 | 1 | 4 |
| 1 | 1 | 1 | 2 |

| $x_4$ | $x_5$ | $\phi_C$ |
|---|---|---|
| 0 | 0 | 8 |
| 1 | 0 | 2 |
| 0 | 1 | 2 |
| 1 | 1 | 6 |

| $x_4$ | $x_6$ | $\phi_D$ |
|---|---|---|
| 0 | 0 | 3 |
| 1 | 0 | 6 |
| 0 | 1 | 6 |
| 1 | 1 | 3 |

(a) Draw the factor graph corresponding to $p(x_2, x_3, x_4, x_5 \mid x_1 = 0, x_6 = 1)$ and give the tables defining the new factors $\phi_A^{x_1=0}(x_2, x_3)$ and $\phi_D^{x_6=1}(x_4)$ that you obtain.

(b) Find $p(x_2 \mid x_1 = 0, x_6 = 1)$ using the elimination ordering $(x_4, x_5, x_3)$:

(i) Draw the graph for $p(x_2, x_3, x_5 \mid x_1 = 0, x_6 = 1)$ by marginalising $x_4$
Compute the table for the new factor $\tilde{\phi}_4(x_2, x_3, x_5)$

(ii) Draw the graph for $p(x_2, x_3 \mid x_1 = 0, x_6 = 1)$ by marginalising $x_5$
Compute the table for the new factor $\tilde{\phi}_{45}(x_2, x_3)$

(iii) Draw the graph for $p(x_2 \mid x_1 = 0, x_6 = 1)$ by marginalising $x_3$
Compute the table for the new factor $\tilde{\phi}_{453}(x_2)$

(c) Note that the previous variable ordering involved computing a new factor $\tilde{\phi}_4$ that depends on three variables $x_2$, $x_3$, and $x_5$, this involved computing $2^3$ numbers (i.e. the rows in the table for $\tilde{\phi}_4$). Instead, now find $p(x_2 \mid x_1 = 0, x_6 = 1)$ using the elimination ordering $(x_5, x_4, x_3)$,

(i) Draw the graph for $p(x_2, x_3, x_4, \mid x_1 = 0, x_6 = 1)$ by marginalising $x_5$
Compute the table for the new factor $\tilde{\phi}_5(x_4)$

(ii) Draw the graph for $p(x_2, x_3 \mid x_1 = 0, x_6 = 1)$ by marginalising $x_4$
Compute the table for the new factor $\tilde{\phi}_{54}(x_2, x_3)$

(iii) Draw the graph for $p(x_2 \mid x_1 = 0, x_6 = 1)$ by marginalising $x_3$
Compute the table for the new factor $\tilde{\phi}_{543}(x_2)$