The purpose of this tutorial sheet is to help you better understand the lecture material. Start early and do as many as you have time for. Even if you are unable to make much progress, you should still attend your tutorial.

## Exercise 1. I-maps

(a) Which of three graphs represent the same set of independencies? Explain.


Graph 1


Graph 2


Graph 3
(b) Let $p(a, z, q, e, h)=p(a) p(z) p(q \mid a, z) p(e \mid q) p(h \mid z)$. In the lecture "Expressive Power of Graphical Models", we have seen that the graph in Figure 1 is a minimal I-map for $p$. Assuming that the graph is a perfect map for $p$, determine minimal directed I-maps for the orderings

- $(a, z, h, q, e)$
- $(e, h, q, z, a)$

Are the two I-maps that you obtain I-equivalent?


Figure 1: Perfect map for Exercise 1, question (b).
(c) For the collection of random variables $(a, z, h, q, e)$ you are given the following Markov blankets for each variable:

- $\operatorname{MB}(\mathrm{a})=\{\mathrm{q}, \mathrm{z}\}$
- $\operatorname{MB}(\mathrm{z})=\{\mathrm{a}, \mathrm{q}, \mathrm{h}\}$
- $\operatorname{MB}(\mathrm{h})=\{\mathrm{z}\}$
- $\operatorname{MB}(\mathrm{q})=\{\mathrm{a}, \mathrm{z}, \mathrm{e}\}$
- $\operatorname{MB}(e)=\{q\}$
(i) Draw the undirected minimal I-map.
(ii) Indicate a Gibbs distribution that satisfies the independence relations specified by the Markov blankets.


## Exercise 2. Conversion between graphs

(a) For distributions that factorises over the graph below, find the minimal undirected I-map.

(b) The graph below is a directed minimal I-map for the hidden Markov model. Find the corresponding undirected minimal I-map.

(c) For the undirected I-map below, what is a corresponding directed minimal I-map?


## Exercise 3. Limits of directed and undirected graphical models

We here consider the probabilistic model $p\left(y_{1}, y_{2}, x_{1}, x_{2}\right)=p\left(y_{1}, y_{2} \mid x_{1}, x_{2}\right) p\left(x_{1}\right) p\left(x_{2}\right)$ where $p\left(y_{1}, y_{2} \mid x_{1}, x_{2}\right)$ factorises as

$$
\begin{equation*}
p\left(y_{1}, y_{2} \mid x_{1}, x_{2}\right)=p\left(y_{1} \mid x_{1}\right) p\left(y_{2} \mid x_{2}\right) \phi\left(y_{1}, y_{2}\right) n\left(x_{1}, x_{2}\right) \tag{1}
\end{equation*}
$$

with $n\left(x_{1}, x_{2}\right)$ equal to

$$
\begin{equation*}
n\left(x_{1}, x_{2}\right)=\left(\int p\left(y_{1} \mid x_{1}\right) p\left(y_{2} \mid x_{2}\right) \phi\left(y_{1}, y_{2}\right) \mathrm{d} y_{1} \mathrm{~d} y_{2}\right)^{-1} \tag{2}
\end{equation*}
$$

In the lecture "Factor Graphs", we used the model to illustrate the setup where $x_{1}$ and $x_{2}$ are two independent inputs that each control the interacting variables $y_{1}$ and $y_{2}$ (see graph below).


In the additional practice questions, you are asked to verify that $p\left(y_{1}, y_{2}, x_{1}, x_{2}\right)$ satisfies the following independencies

$$
x_{1} \Perp x_{2} \quad x_{1} \Perp y_{2}\left|y_{1}, x_{2} \quad x_{2} \Perp y_{1}\right| y_{2}, x_{1}
$$

We here investigate whether undirected and directed graph can represent these independencies.
(a) Draw the undirected graph for $p\left(y_{1}, y_{2}, x_{1}, x_{2}\right)$ and check whether graph separation allows us to see all independencies listed above.
(b) Draw the directed graph for $p\left(y_{1}, y_{2}, x_{1}, x_{2}\right)=p\left(y_{1}, y_{2} \mid x_{1}, x_{2}\right) p\left(x_{1}\right) p\left(x_{2}\right)$ and check whether graph separation allows us to see all independencies listed above.

