The purpose of this additional sheet is to provide more practice and exam preparation material. N.B. The tutors are not required to work through this material in the tutorial.

Exercise 1. ***Restricted Boltzmann machine (based on Barber Exercise 4.4)***

The restricted Boltzmann machine is an undirected graphical model for binary variables $\mathbf{v} = (v_1, \ldots, v_n)^\top$ and $\mathbf{h} = (h_1, \ldots, h_m)^\top$ with a probability mass function equal to

$$p(\mathbf{v}, \mathbf{h}) \propto \exp\left(\mathbf{v}^\top \mathbf{W}\mathbf{h} + \mathbf{a}^\top \mathbf{v} + \mathbf{b}^\top \mathbf{h}\right), \tag{1}$$

where $\mathbf{W}$ is a $n \times m$ matrix. Both the v_i and h_i take values in $\{0, 1\}$. The v_i are called the "visibles" variables since they are assumed to be observed while the h_i are the hidden variables since it is assumed that we cannot measure them.

(a) Use graph separation to show that the joint conditional $p(\mathbf{h}|\mathbf{v})$ factorises as

$$p(\mathbf{h}|\mathbf{v}) = \prod_{i=1}^{m} p(h_i|\mathbf{v}).$$

(b) Show that

$$p(h_i = 1|\mathbf{v}) = \frac{1}{1 + \exp\left(-b_i - \sum_j W_{ji} v_j\right)} \tag{2}$$

where W_{ji} is the (ji)-th element of $\mathbf{W}$, so that $\sum_j W_{ji} v_j$ is the inner product (scalar product) between the i-th column of $\mathbf{W}$ and $\mathbf{v}$.

(c) Use a symmetry argument to show that

$$p(\mathbf{v}|\mathbf{h}) = \prod_i p(v_i|\mathbf{h}) \quad \text{and} \quad p(v_i = 1|\mathbf{h}) = \frac{1}{1 + \exp\left(-a_i - \sum_j W_{ij} h_j\right)}$$