# Expressive Power of Graphical Models
# — Supplement —

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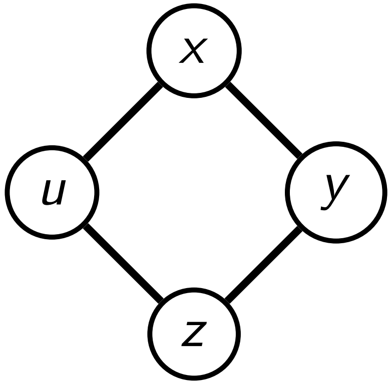
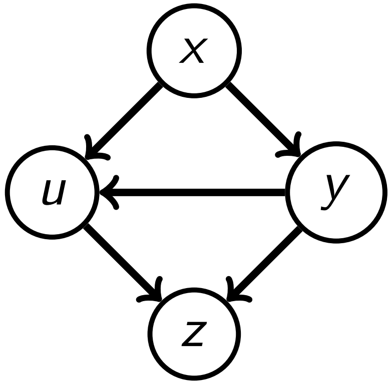
Goal: Given undirected I-map $H$, find directed minimal I-map $G$ for $\mathcal{I}(H)$. In other words, find a DAG $G$ such that $\mathcal{I}(G) \subseteq \mathcal{I}(H)$.




Given: undirected I-map $H$

$$x \perp\!\!\!\perp z \mid u, y$$
$$u \perp\!\!\!\perp y \mid x, z$$




Solution: directed minimal I-map $G$

(with ordering: $x, y, u, z$)

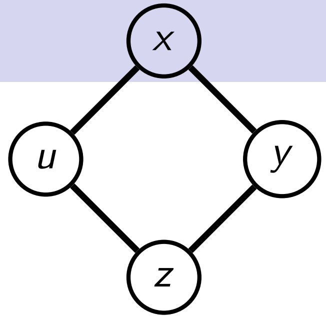
$$x \perp\!\!\!\perp z \mid u, y$$
$$u \not\perp\!\!\!\perp y \mid x, z$$

Note: We lost information with the conversion. There is no DAG $G$ with $\mathcal{I}(G) = \{x \perp\!\!\!\perp z \mid u, y \quad u \perp\!\!\!\perp y \mid x, z\}$

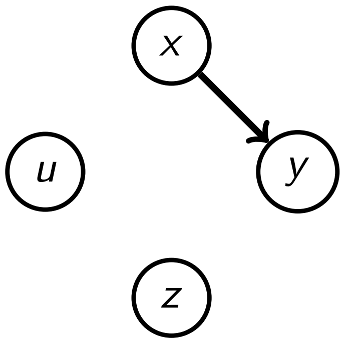
# Procedure

- In order to construct the directed minimal I-map, we proceed as in slide 12 of the"Expressive Power of Graphical Models" slides with the small modification that we check whether $x_i \perp\!\!\!\perp \{\mathrm{pre}_i \setminus \pi_i\} \mid \pi_i$ is included in $\mathcal{I}(H)$ rather than in $\mathcal{I}(p)$. That is:
  1. Assume an ordering of the variables. Denote the ordered random variables by $x_1, \ldots, x_d$.
  2. For each $i$, find a minimal subset of variables $\pi_i \subseteq \mathrm{pre}_i$ such that
  $$x_i \perp\!\!\!\perp \{\mathrm{pre}_i \setminus \pi_i\} \mid \pi_i$$
  is an element of $\mathcal{I}(H)$ (i.e. the independency is asserted by $H$).
  3. Construct a graph with parents $\mathrm{pa}_i = \pi_i$.
- We next derive the solution for the example with the indicated ordering $x, y, u, z$

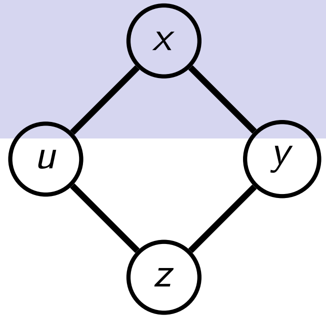
# Step 1: variable $x$ is the root; consider 2nd variable $y$



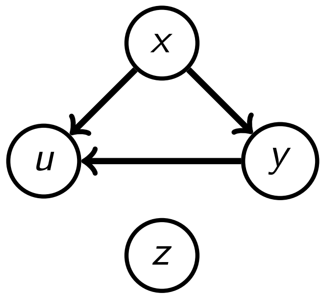

- $\mathrm{pre}(y) = \{x\}$.
- Since $x$ and $y$ are connected in the undirected graph, there must also be an edge in the directed graph; otherwise the directed graph would make a wrong independence assertion.
- (In more detail, if we didn't have an edge in $G$, the graph would assert that $x$ and $y$ are independent. But since this independency is not included in $\mathcal{I}(H)$, we must have an edge $x \to y$ in $G$ in order for $G$ to be an I-map.)
- We thus obtain:



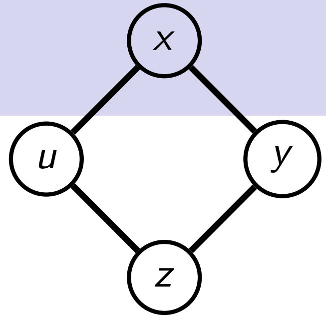
# Step 2: consider $u$ next




- $\mathrm{pre}(u) = \{x, y\}$.
- Since we want a *minimal* subset $\pi_u$ of $\mathrm{pre}(u)$, let us first try $\pi_u = \varnothing$. If $\pi_u = \varnothing$ held, the directed graph would assert $u \perp\!\!\!\perp \{x, y\}$ (by the ordered Markov property). We thus have to check whether the undirected graph $H$ makes this assertion too. Since $x$ and $u$ are connected in the undirected graph, $u \perp\!\!\!\perp \{x, y\} \notin \mathcal{I}(H)$, and we thus cannot set $\mathrm{pa}_u = \varnothing$.
- We next try out singleton sets: if $\pi_u = \{x\}$, $\mathrm{pre}(u) \setminus \pi_u = \{y\}$, and the directed graph would assert $u \perp\!\!\!\perp y \mid x$. But since this independency is not asserted by $h$, $\{x\}$ is not the desired subset. The same reasoning shows that $\pi_u = \{y\}$ does not work either.
- We thus have to set $\pi_u = \mathrm{pre}(u) = \{x, y\}$ and obtain



## Step 3: consider $z$ next (last variable in the ordering)




- $\mathrm{pre}(z) = \{x, y, u\}$.
- We could proceed as in step 2 to find the minimal subset $\pi_z \subseteq \mathrm{pre}(z)$ such that $z \perp\!\!\!\perp \{\mathrm{pre}(z) \setminus \pi_z\} \mid \pi_z$. However, since $\mathrm{pre}(z)$ corresponds to all nodes in the graph (without $z$), the desired $\pi_z$ is exactly the Markov blanket of $z$.
- We can thus use the rules to determine the Markov blanket to determine $\pi_z$ (which is faster than proceeding as in step 2).
- From $H$, we find that $\mathrm{MB}(z) = \mathrm{ne}(z) = \{u, y\}$. Hence: $\pi_z = \{u, y\}$ and we obtain: