THE UNIVERSITY *of* EDINBURGH **informatics**

Probabilistic Modelling and Reasoning
Tutorial 3

Spring 2018
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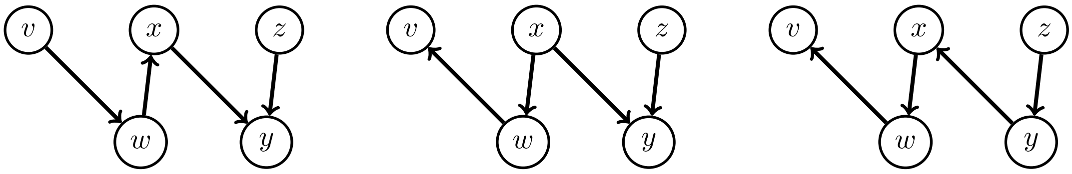
The purpose of the tutorials is twofold: First, they help you better understand the lecture material. Secondly, they provide exam preparation material. You are not expected to complete all questions before the tutorial sessions. Start early and do as many as you have time for.

Exercise 1. *I-maps*

(a) Which of three graphs represent the same set of independencies? Explain.



Graph 1 Graph 2 Graph 3

(b) For $p(a, z, q, e, h) = p(a)p(z)p(q|a, z)p(e|q)p(h|z)$, determine a minimal I-map for the orderings

- (a, z, q, e, h)
- (a, z, h, q, e)
- (e, h, q, z, a)

Are the I-maps I-equivalent?

(c) For the collection of random variables (a, z, h, q, e) you are given the following Markov blankets for each variable:

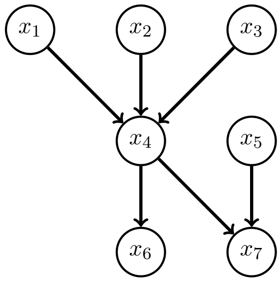
- MB(a) = {q,z}
- MB(z) = {a,q,h}
- MB(h) = {z}
- MB(q) = {a,z,e}
- MB(e) = {q}

(i) Draw the undirected minimal I-map.

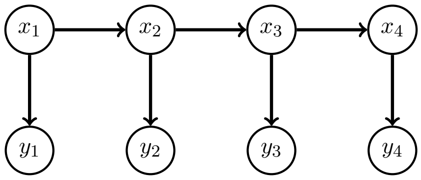
(ii) Indicate a Gibbs distribution that satisfies the independence relations specified by the Markov blankets.

Exercise 2. *Conversion between graphs*

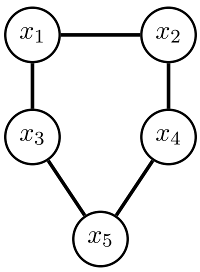
(a) For distributions that factorises over the graph below, find the minimal undirected I-map.



(b) The graph below is a directed minimal I-map for the hidden Markov model. Find the corresponding undirected minimal I-map.

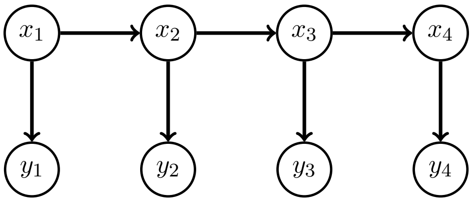


(c) For the undirected I-map below, what is a corresponding directed minimal I-map?

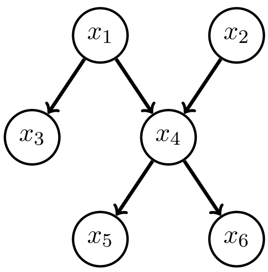


(d) Draw an undirected graph and an undirected factor graph for $p(x_1, x_2, x_3) = p(x_1)p(x_2)p(x_3|x_1, x_2)$

(e) Draw an undirected factor graph for the directed graphical model defined by the graph below.



(f) Draw the moralised graph and an undirected factor graph for directed graphical models defined by the graph below (this kind of graph is called a polytree: there are no loops but a node may have more than one parent).



Exercise 3. ***Limits of directed and undirected graphical models***

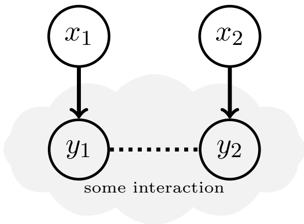
We here consider the probabilistic model $p(y_1, y_2, x_1, x_2) = p(y_1, y_2|x_1, x_2)p(x_1)p(x_2)$ where $p(y_1, y_2|x_1, x_2)$ factorises as

$$p(y_1, y_2|x_1, x_2) = p(y_1|x_1)p(y_2|x_2)\phi(y_1, y_2)n(x_1, x_2) \tag{1}$$

with $n(x_1, x_2)$ equal to

$$n(x_1, x_2) = \left(\int p(y_1|x_1)p(y_2|x_2)\phi(y_1, y_2)\mathrm{d}y_1\mathrm{d}y_2 \right)^{-1}. \tag{2}$$

In the lecture, we used the model to illustrate the setup where x_1 and x_2 are two independent inputs that each control the interacting variables y_1 and y_2 (see graph below).



(a) Use the basic characterisations of statistical independence

$$u \perp\!\!\!\perp v|z \Longleftrightarrow p(u, v|z) = p(u|z)p(v|z) \tag{3}$$
$$u \perp\!\!\!\perp v|z \Longleftrightarrow p(u, v|z) = a(u, z)b(v, z) \qquad (a(u, z) \geq 0, b(v, z) \geq 0) \tag{4}$$

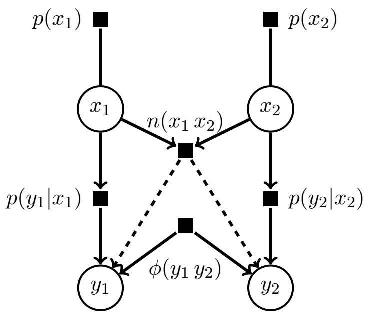
to show that $p(y_1, y_2, x_1, x_2)$ satisfies the following independencies

- $x_1 \perp\!\!\!\perp x_2$ (independence between control variables)
- $x_1 \perp\!\!\!\perp y_2 \mid y_1, x_2$ (y_2 is only influenced by y_1 and x_2)
- $x_2 \perp\!\!\!\perp y_1 \mid y_2, x_1$ (y_1 is only influenced by y_2 and x_1)

(b) Draw the undirected graph for $p(y_1, y_2, x_1, x_2)$ and check whether graph separation allows us to see all independencies listed above.

(c) Draw the directed graph for $p(y_1, y_2, x_1, x_2) = p(y_1, y_2|x_1, x_2)p(x_1)p(x_2)$ and check whether graph separation allows us to see all independencies listed above.

(d) *(optional, not examinable)* In the lecture, we have the following factor graph for $p(y_1, y_2, x_1, x_2)$



Use the separation rules for factor graphs to verify that we can find all independence relations. The separation rules are (see Barber, section 4.4.1, or the original paper by Brendan Frey: https://arxiv.org/abs/1212.2486):

"If all paths are blocked, the variables are conditionally independent. A path is blocked if one or more of the following conditions is satisfied:

1. One of the variables in the path is in the conditioning set.
2. One of the variables or factors in the path has two incoming edges that are part of the path (variable or factor collider), and neither the variable or factor nor any of its descendants are in the conditioning set."

Remarks:

- "one or more of the following" should best be read as "one of the following".
- "incoming edges" means directed incoming edges
- the descendants of a variable of factor node are all the variables that you can reach by following a path (containing directed or directed edges, but for directed edges, all directions have to be consistent)
- In the graph we have dashed directed edges: they do count when you determine the descendants but they do not contribute to paths. For example, y_1 is a descendant of the $n(x_1, x_2)$ factor node but $x_1 - n - y_2$ is not a path.