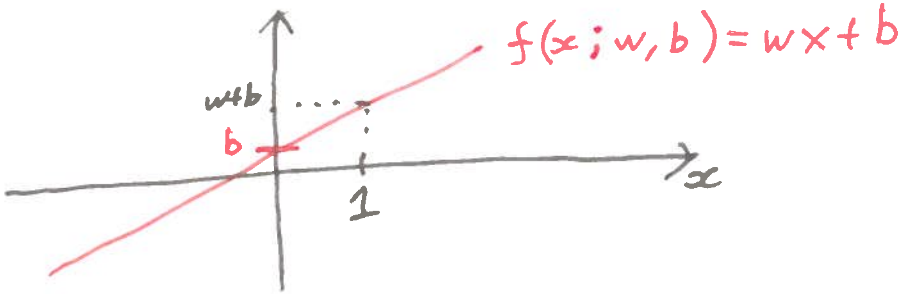
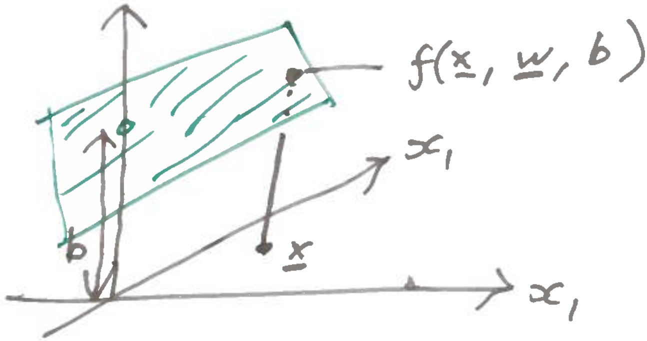
Linear Functions

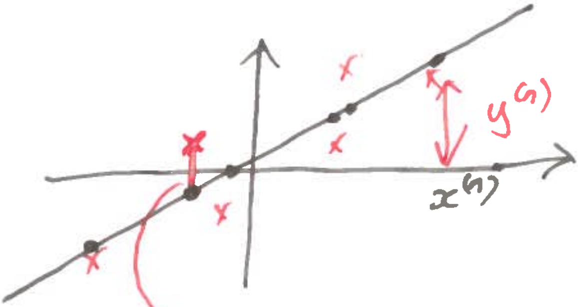




$$f(\underline{x}; \underline{w}, b) = w_1 x_1 + w_2 x_2 + b$$

$$= \underline{w}^T \underline{x} + b$$

Data $\{(\underline{x}^{(n)}, y^{(n)})\}_{n=1}^{N}$



Residual $y^{(n)} - f(x^{(n)}; w, b)$

In vector and matrix format:

$$\underline{y} = \begin{bmatrix} y^{(1)} \\ y^{(2)} \\ \vdots \\ y^{(N)} \end{bmatrix} \qquad X = \begin{bmatrix} - & \underline{x}^{(1)T} & - \\ - & \underline{x}^{(2)T} & - \\ & \vdots & \\ x_1^{(N)} & \cdots & x_D^{(N)} \end{bmatrix}$$

N×1 matrix

D-dimensional regression.

Python: y is (N,) vector
Then y[:, None] is (N,1) array
(Really NumPy)

$$\underline{f} = \begin{bmatrix} f(\underline{x}^{(1)}; \underline{w}, b) \\ \vdots \\ f(\underline{x}^{(N)}; \underline{w}, b) \end{bmatrix}$$

Least squares fitting

Minimize $\displaystyle\sum_{n=1}^{N} \left(y^{(n)} - f(x^{(n)}; \underline{w}, b)\right)^2$

Minimize $(\underline{y} - \underline{f})^T (\underline{y} - \underline{f})$

Models with zero intercept $(b=0)$

$f(\underline{x}; \underline{w}) = \underline{w}^T \underline{x} = \underline{x}^T \underline{w}$

$$\underset{N\times 1}{\underline{f}} = \underset{N\times D}{X} \underset{D\times 1}{\underline{w}} \approx \underline{y}$$

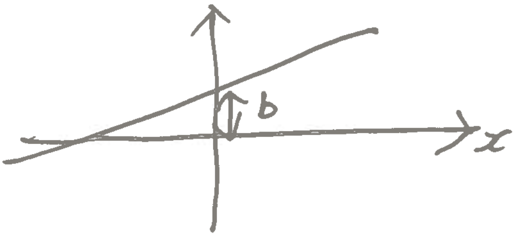
"Linear map"

$g(\underline{x} + \underline{z}) = g(\underline{x}) + g(\underline{z})$

$g(c\underline{x}) = c\, g(\underline{x})$

Matlab: `w_fit = X \ y;`

Python `= np.linalg.lstsq(X, y)[0]`



$$X' = \begin{bmatrix} 1 & -\underline{x}^{(1)T}- \\ 1 & -\underline{x}^{(2)T}- \\ \vdots & \vdots \\ 1 & -\underline{x}^{(N)T}- \end{bmatrix}$$

$$\underline{w}' = \operatorname{argmin} \| \underline{y} - X'\underline{w}' \|^2$$

Fits $\underline{y}$ with $\underline{f} = X'\underline{w}' = w_1' + X w'_{2:D+1}$

$$= b + X\underline{w}$$

$$\underline{f} = \Phi\, \underline{w}$$

Any representation we like

Each row is a datapoint

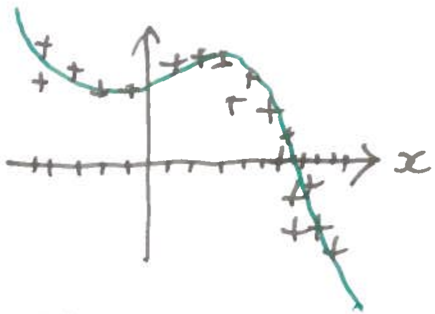
$$\Phi = \begin{bmatrix} — \ \underline{\phi}(\underline{x}^{(1)})^T \ — \\ \vdots \\ — \ \underline{\phi}(\underline{x}^{(N)})^T \ — \end{bmatrix}$$

<u>Example</u>

$$\underline{\phi}(x) = [1 \quad x \quad x^2 \quad x^3]^T$$

Fit $\underline{y} \approx \underline{f} = \Phi\, \underline{w}$ (by "linear regression")

$$f(x) = w_1 + w_2 x + w_3 x^2 + w_4 x^3$$



cubic polynomial fit