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Tutorial Exercises for fifth week

1. Use the inference system given below.You may construct the derivations top-down or bottom-up if you find that more natural.

   The syntax uses the notion of sequent, *ie* the statements at any point in the derivation take the form a *sequent*

   $$\texttt{Set\_of\_Formulas} \Longrightarrow \texttt{Set\_of\_Formulas}$$

   indicating that *some* formula on the right follows from *all* of the assumptions of the left.

   Derivations take the form of a numbered list of such assertions, labeled with a justification. The justification is one of:

   - That the assertion is an axiom.
   - That the assertion follows from previous assertions by an inference rule.

   There is also a modal operator, $\Box$, built in to the logic (so this is a *modal logic*). $\Box \mathtt{P}$ is interpreted as "*P* is necessarily true".

   There are two rules for each connective, one when it appears on the right, and one when it appears on the left of the sequent (in the conclusion).

Axioms and Rules of Inference

(axiom) $\mathtt{G}, \mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \mathtt{G}, \mathtt{G}_1, \ldots, \mathtt{G}_m$ (Axiom)

$$\text{(andI)}\ \frac{\mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \mathtt{F}, \mathtt{G}_1, \ldots, \mathtt{G}_m \qquad \mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \mathtt{G}, \mathtt{G}_1, \ldots, \mathtt{G}_m}{\mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \mathtt{F} \land \mathtt{G}, \mathtt{G}_1, \ldots, \mathtt{G}_m}$$

$$\text{(andE)}\ \frac{\mathtt{F}, \mathtt{G}, \mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \mathtt{C}}{\mathtt{F} \land \mathtt{G}, \mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \mathtt{C}}$$

$$\text{(impI)}\ \frac{\mathtt{F}, \mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \mathtt{G}, \mathtt{G}_1, \ldots, \mathtt{G}_m}{\mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \mathtt{F} \to \mathtt{G}, \mathtt{G}_1, \ldots, \mathtt{G}_m}$$

$$\text{(impE)}\ \frac{\mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \mathtt{F}, \mathtt{C}_1, \ldots, \mathtt{C}_m \qquad \mathtt{G}, \mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \mathtt{C}_1, \ldots, \mathtt{C}_m}{\mathtt{F} \to \mathtt{G}, \mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \mathtt{C}_1, \ldots, \mathtt{C}_m}$$

$$\text{(orI)}\ \frac{\mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \mathtt{F}, \mathtt{G}, \mathtt{G}_1, \ldots, \mathtt{G}_m}{\mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \mathtt{F} \lor \mathtt{G}, \mathtt{G}_1, \ldots, \mathtt{G}_m}$$

$$\text{(orE)}\ \frac{\mathtt{F}, \mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \mathtt{C}_1, \ldots, \mathtt{C}_m \qquad \mathtt{G}, \mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \mathtt{C}_1, \ldots, \mathtt{C}_m}{\mathtt{F} \lor \mathtt{G}, \mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \mathtt{C}_1, \ldots, \mathtt{C}_m}$$

$$(\text{notI})\ \frac{\mathtt{F}, \mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \mathtt{C}_1, \ldots, \mathtt{C}_m}{\mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \neg\mathtt{F}, \mathtt{C}_1, \ldots, \mathtt{C}_m}$$

$$(\text{notE})\ \frac{\mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \mathtt{F}, \mathtt{G}_1, \ldots, \mathtt{G}_m}{\neg\mathtt{F}, \mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \mathtt{G}_1, \ldots, \mathtt{G}_m}$$

$$(\text{boxI})\ \frac{\mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \mathtt{G}, \mathtt{P}_1, \ldots, \mathtt{P}_m}{\Box\mathtt{F}_1, \ldots \Box\mathtt{F}_n, \mathtt{G}_1 \ldots \mathtt{G}_m \Longrightarrow \Box\mathtt{G}, \Box\mathtt{P}_1, \ldots \Box\mathtt{P}_m, \mathtt{H}_1, \ldots, \mathtt{H}_k}$$

$$(\text{boxE})\ \frac{\mathtt{F}, \mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \mathtt{H}_1, \ldots, \mathtt{H}_k}{\Box\mathtt{F}, \mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \mathtt{H}_1, \ldots, \mathtt{H}_k}$$

2. The following are *not* derivable in this system; use the system to construct counter-models for the formulas (i.e. find an assignment of truth values to the propositional symbols that makes the statement false, by building a failed proof attempt in the system above).

   (a) $\neg a \rightarrow (b \rightarrow a)$

   (b) $((a \wedge b) \rightarrow c) \rightarrow ((a \vee b) \rightarrow c)$

3. For each of the following, use the semantics of first-order logic to show that the proposed inference rule is sound, or unsound.

   (a) $\dfrac{\mathtt{X} \rightarrow \neg\mathtt{Y}}{\mathtt{Y} \rightarrow \neg\mathtt{X}}$

   (b) $\dfrac{\neg\mathtt{X} \rightarrow \neg\mathtt{Y}}{\mathtt{X} \rightarrow \mathtt{Y}}$

   (c) $\dfrac{\neg\forall x\ P(x)}{\exists x\ \neg P(x)}$

   (d) $\dfrac{\forall x\ \forall y\ Q(x, y)}{\forall y\ \forall x\ Q(x, y)}$

   (e) $\dfrac{\forall x\ \exists y\ Q(x, y)}{\exists y\ \forall x\ Q(x, y)}$