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Tutorial Exercises for third week

1. For each of the following, give an interpretation to the non-logical symbols such that the sentence makes sense, and that represents the world accurately (*ie* you believe it to be true).

   (a) $\texttt{2} > \texttt{3}$

   (b) $\forall \texttt{x} \; ((\neg \texttt{foo}(\texttt{x})) \rightarrow (\neg \texttt{baz}(\texttt{x})))$

   (c) $\forall \texttt{x} \; \forall \texttt{y} \; (\texttt{p}(\texttt{x}, \texttt{y}) \rightarrow \texttt{p}(\texttt{y}, \texttt{x}))$

2. Give a model (*ie* a structure and an interpretation) that makes the first formula true and the second formula false. (Here $\texttt{x}, \texttt{y}, \texttt{z}$ are variables.)

   (a) $\forall \texttt{x} \; \forall \texttt{y} \; \forall \texttt{z} \; ((\texttt{p}(\texttt{x}, \texttt{y}) \wedge \texttt{p}(\texttt{y}, \texttt{z})) \rightarrow \texttt{p}(\texttt{x}, \texttt{z}))$

   (b) $\forall \texttt{x} \; \forall \texttt{y} \; ((\texttt{p}(\texttt{x}, \texttt{y}) \wedge \texttt{p}(\texttt{y}, \texttt{x})) \rightarrow \texttt{x} = \texttt{y})$

3. Use the inference system given below.You may construct the derivations top-down or bottom-up if you find that more natural.

   The syntax uses the notion of sequent, *ie* the statements at any point in the derivation take the form a *sequent*

   $$\texttt{Set\_of\_Formulas} \Longrightarrow \texttt{Set\_of\_Formulas}$$

   indicating that *some* formula on the right follows from *all* of the assumptions of the left.

   Derivations take the form of a numbered list of such assertions, labeled with a justification. The justification is one of:

   - That the assertion is an axiom.
   - That the assertion follows from previous assertions by an inference rule.

   There is also a modal operator, $\Box$, built in to the logic (so this is a *modal logic*). $\Box \texttt{P}$ is interpreted as "P is necessarily true".

   There are two rules for each connective, one when it appears on the right, and one when it appears on the left of the sequent (in the conclusion).

Axioms and Rules of Inference

(axiom) $\mathtt{G}, \mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \mathtt{G}, \mathtt{G}_1, \ldots, \mathtt{G}_m$ (Axiom)

(andI) $$\frac{\mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \mathtt{F}, \mathtt{G}_1, \ldots, \mathtt{G}_m \qquad \mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \mathtt{G}, \mathtt{G}_1, \ldots, \mathtt{G}_m}{\mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \mathtt{F} \land \mathtt{G}, \mathtt{G}_1, \ldots, \mathtt{G}_m}$$

(andE) $$\frac{\mathtt{F}, \mathtt{G}, \mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \mathtt{C}}{\mathtt{F} \land \mathtt{G}, \mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \mathtt{C}}$$

(impI) $$\frac{\mathtt{F}, \mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \mathtt{G}, \mathtt{G}_1, \ldots, \mathtt{G}_m}{\mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \mathtt{F} \to \mathtt{G}, \mathtt{G}_1, \ldots, \mathtt{G}_m}$$

(impE) $$\frac{\mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \mathtt{F}, \mathtt{C}_1, \ldots, \mathtt{C}_m \qquad \mathtt{G}, \mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \mathtt{C}_1, \ldots, \mathtt{C}_m}{\mathtt{F} \to \mathtt{G}, \mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \mathtt{C}_1, \ldots, \mathtt{C}_m}$$

(orI) $$\frac{\mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \mathtt{F}, \mathtt{G}, \mathtt{G}_1, \ldots, \mathtt{G}_m}{\mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \mathtt{F} \lor \mathtt{G}, \mathtt{G}_1, \ldots, \mathtt{G}_m}$$

(orE) $$\frac{\mathtt{F}, \mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \mathtt{C}_1, \ldots, \mathtt{C}_m \qquad \mathtt{G}, \mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \mathtt{C}_1, \ldots, \mathtt{C}_m}{\mathtt{F} \lor \mathtt{G}, \mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \mathtt{C}_1, \ldots, \mathtt{C}_m}$$

(notI) $$\frac{\mathtt{F}, \mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \mathtt{C}_1, \ldots, \mathtt{C}_m}{\mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \neg\mathtt{F}, \mathtt{C}_1, \ldots, \mathtt{C}_m}$$

(notE) $$\frac{\mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \mathtt{F}, \mathtt{G}_1, \ldots, \mathtt{G}_m}{\neg\mathtt{F}, \mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \mathtt{G}_1, \ldots, \mathtt{G}_m}$$

(boxI) $$\frac{\mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \mathtt{G}, \mathtt{P}_1, \ldots, \mathtt{P}_m}{\Box\mathtt{F}_1, \ldots \Box\mathtt{F}_n, \mathtt{G}_1 \ldots \mathtt{G}_m \Longrightarrow \Box\mathtt{G}, \Box\mathtt{P}_1, \ldots \Box\mathtt{P}_m, \mathtt{H}_1, \ldots, \mathtt{H}_k}$$

(boxE) $$\frac{\mathtt{F}, \mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \mathtt{H}_1, \ldots, \mathtt{H}_k}{\Box\mathtt{F}, \mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \mathtt{H}_1, \ldots, \mathtt{H}_k}$$

(a) Give derivations for the following (ie proofs of the formula from an empty hypothesis list).

i. $(\mathsf{p} \lor (\mathsf{q} \lor \mathsf{r})) \to ((\mathsf{p} \lor \mathsf{q}) \lor \mathsf{r})$

ii. $\neg\mathsf{p} \to (\mathsf{p} \to \mathsf{q})$

iii. $(\mathsf{p} \to (\mathsf{q} \to \mathsf{r})) \to ((\mathsf{p} \to \mathsf{q}) \to (\mathsf{p} \to \mathsf{r}))$

(b) Give derivations for the following modal statements.

i. $(\Box\Box\mathsf{p}) \to ((\Box(\mathsf{p} \to \mathsf{q})) \to \Box\mathsf{q})$

ii. $(\neg(\Box(\mathsf{p} \land \mathsf{q}))) \to ((\neg(\Box\mathsf{p})) \lor (\neg(\Box\mathsf{q})))$