Today

Logic and Search ctd

Recall: Logical Consequence

Our semantics gives us a notion of *logical consequence*.

We say that a formula G is a logical consequence of formulae $F_1, F_2, \ldots, F_n$ (meaning that it follows logically) if and only if, for all structures with interpretation $\mathcal{S}$,

$$\text{if } \mathcal{S} \models F_1 \text{ and } \ldots \text{ and } \mathcal{S} \models F_n, \\ \text{then } \mathcal{S} \models G.$$

When this is true, we write

$$F_1, F_2, \ldots F_n \models G.$$

Relating proof and semantics

We can use our semantics to check whether an inference rule is *sound* or not.

We say an inference rule

$$\frac{A_1 \ldots A_n}{B}$$

is *sound* if for every set of formulae $F_1 \ldots F_n, G$ that match the inference rule, we have

$$F_1, \ldots, F_n \models G$$

For example, the rule

$$\frac{P \qquad P \rightarrow Q}{Q}$$

is sound. An inference system is sound if all the inference rules are sound, and all the axioms are valid.

An example

To show that MP is sound, we can just use truth tables, which are built into the semantics of $\wedge, \rightarrow, \vee, \neg$.

We want to show that if $\mathcal{S} \models A \rightarrow B$ and $\mathcal{S} \models A$, then $\mathcal{S} \models B$.

This tells us that A is true in S, and $A \rightarrow B$ is also true. Looking at the truth table for $\rightarrow$, we see that B must be true as well.

So the inference rule is sound.

An unsound inference rule

Suppose a rule is proposed:

$$\frac{\neg \mathtt{P} \qquad \mathtt{P} \rightarrow \mathtt{Q}}{\neg \mathtt{Q}}$$

This is in fact unsound. To show this, we need to find interpretations and formulas that make the formulas above the line true, and the conclusion false.

For example, take P as $\neg(\mathsf{a} \rightarrow \mathsf{a})$, and Q as $\mathsf{a} \rightarrow \mathsf{a}$. Now check:

$\neg\neg(\mathsf{a} \rightarrow \mathsf{a})$	is *true* (for any S)
$(\neg(\mathsf{a} \rightarrow \mathsf{a})) \rightarrow (\mathsf{a} \rightarrow \mathsf{a})$	is *true* (for any S)
$\neg(\mathsf{a} \rightarrow \mathsf{a})$	is *false* (for any S)

So the inference rule is *unsound*.

Sequent rules

Remember that we understand a sequent

$$F_1, F_2, \ldots, F_n \Longrightarrow G_1, \ldots, G_m$$

as saying: if all the lhs formulas are true, then at least one of the rhs formulas is true.

To show that a sequent rule is sound, say for

$$\frac{\mathtt{LForms} \Longrightarrow \mathtt{RForms}}{\mathtt{LForms}' \Longrightarrow \mathtt{RForms}'}$$

we need to show that whenever $\mathtt{LForms} \Longrightarrow \mathtt{RForms}$ is true, so is $\mathtt{LForms}' \Longrightarrow \mathtt{RForms}'$.

Example: NotI is sound

The rule is:

$$\frac{\mathtt{F}, \mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \mathtt{C}_1, \ldots, \mathtt{C}_m}{\mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \neg\mathtt{F}, \mathtt{C}_1, \ldots, \mathtt{C}_m}$$

We are only interest in the cases where the top sequent is true. Look at the cases:

1. One of $\mathtt{F}_1, \ldots, \mathtt{F}_n$ is false; then the sequent below is true.
2. F is false; then conclusion is true, since a rhs formula is true.
3. One of $\mathtt{C}_1, \ldots, \mathtt{C}_m$ is true; then the conslusion is true.

The rule is sound, since there are no more cases where the sequent above is true.

Proof search in sequent systems

We have looked at finding proofs in sequent systems using backward search from the goal statement. There is choice in how the search is carried out; in general we build partial trees, which we can extend at the leaves. Nodes in the search space are connected by a single rule application.

Suppose we search in this way, using the inference rules of the handout. There are three possibilities:

- We may end with a derivation (every leaf is an axiom).
- We may keep applying the rules for ever.
- We arrive at a position where no rules apply, but some of the leaves are not axioms — maybe we should backtrack?

Propositional Logic

Let's think about the propositional rules (with $\wedge, \rightarrow, \vee, \rightarrow, \neg$).

There are two very good properties of this system:

1. The search will always terminate.

2. There is no need to backtrack.

Termination

Claim: Given any sequent S, there is a bound on the size of the tree that can be built by any series of backward applications of the inference rules. Thus any non-backtracking search will terminate with a proof, or with a tree with a leaf which is not an axiom.

Justification: In every inference rule, in each sequent above the line there is 1 fewer connective than in the sequent below the line. Thus each branch in the tree cannot be longer than length n, where n is the number of connectives in the sequent. In the worst base, each node has two branches from it. So the total number of nodes is not more than 2^n.

Backtracking not needed

If we look at the rules, they have a special property:
if one of the sequents above the line is false (in some interpretation), then the conclusion is false (in the same interpretation).

There is no reason for this to be true in general!!

Example: impl:

$$\frac{\mathtt{F}, \mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \mathtt{G}, \mathtt{G}_1, \ldots, \mathtt{G}_m}{\mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \mathtt{F} \rightarrow \mathtt{G}, \mathtt{G}_1, \ldots, \mathtt{G}_m}$$

For the sequent above to be false, we have $\mathtt{F}, \mathtt{F}_1, \ldots, \mathtt{F}_n$ all true, and $\mathtt{G}, \mathtt{G}_1, \ldots, \mathtt{G}_m$ all false; but then $\mathtt{F}_1, \ldots, \mathtt{F}_n$ are all true, $\mathtt{G}_1, \ldots, \mathtt{G}_m$ all false, and $\mathtt{F} \rightarrow \mathtt{G}$ false; so the sequent below is false.

Call this property *invertibility*.

Counter-models

A counter-model for a formula (or sequent) is just an interpretation of the symbols that makes the statement false (for propositions, an assignment of truth values).

If all the rules are invertible, then we can read off a counter-model to a statement by looking at the leaves of a failed proof attempt.

Find a leaf that is not an axiom; make it false by assigning all the atoms on the lhs "true", and all the atoms on the rhs "false". Any other atoms can be assigned any values.

Now by using invertibility repeatedly along the branch to the goal, we see that the goal formula/sequent has a counter-model too.

Thus there is no point in backtracking and looking elsewhere for a derivation.

Example

Let's look at $\Longrightarrow (p \to q) \to ((\neg p) \to q)$. A proof attempt ends up with:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Longrightarrow p,p,q}\,??? \qquad \dfrac{}{q \Longrightarrow p,q}\,axiom}{p \to q \Longrightarrow p,q}\,impE}{\neg p, p \to q \Longrightarrow q}\,notE}{p \to q \Longrightarrow \neg p \to q}\,impI}{\Longrightarrow (p \to q) \to (\neg p \to q)}\,impI$$

To make the non-axiom leaf false, we put p, q both false; now you can check that all the sequents on that branch are false, including $\Longrightarrow (p \to q) \to (\neg p \to q)$.

Search in modal sequent systems

Consider the system with BoxI and BoxE rules from the hand-out.

(boxI) $$\frac{F_1, \ldots, F_n \Longrightarrow G, P_1, \ldots, P_m}{\Box F_1, \ldots \Box F_n, G_1 \ldots G_m \Longrightarrow \Box G, \Box P_1, \ldots \Box P_m, H_1, \ldots, H_k}$$

(boxE) $$\frac{F, F_1, \ldots, F_n \Longrightarrow H_1, \ldots, H_k}{\Box F, F_1, \ldots, F_n \Longrightarrow H_1, \ldots, H_k}$$

In the boxI rule, there must be a box formula on the right in the conclusion ($\Box G$), and there may be other box formulas as indicated.

What can we say about the extended system?

Termination

We can use the same argument as before to show that the search space is finite.

Just check the two new rules; in each case there are fewer connectives above the line than below, and no branching here. So the same upper bound still applies: 2^n where n is the number of connectives.

Backtracking

We cannot use the same argument as before (we haven't said what a model of a modal statement looks like).

Consider the goal

$$\Box\Box p \Longrightarrow \Box(p \wedge p)$$

There is a choice of rules to apply.
If we use BoxE on the $\Box\Box p$, we get the goal

$$\Box p \Longrightarrow \Box(p \wedge p)$$

another BoxE gives us

$$p \Longrightarrow \Box(p \wedge p)$$

There is no derivation for this (even though a different sequence of rules does lead us to a derivation).

What to do?

- Since the search space is finite, we can search it exhaustively – it doesn't matter what strategy we use (depth-first, breadth-first etc).
- Maybe we can figure out a good choice of precedence for the rules (always use BoxI rather than BoxE when both are applicable?).

  Now we need to check that we have not lost completeness – are there any derivable sequents which do not have a derivation under this condition?

Summary

- Semantics to show soundness of inference rules
- Counter-models to show non-derivability
- Good properties of propositional sequent calculus
- More complicated with modal logic