# Today

**Modal Logics**
necessity, possibility, knowledge, belief . . .





Some arguments go easily from natural language to the predicate calculus.

All men are mortal. Fred is a man. Therefore, Fred is mortal.

This corresponds to a derivation in the predicate calculus of

$$\begin{array}{rl} & \forall x\ man(x) \rightarrow mortal(x) \\ & man(fred) \\ \vdash & mortal(fred) \end{array}$$

Other notions are not so easily expressed in terms of truth. *Modal logic* allows formulas to express different *modes* of assertion, beyond just true and false.





# An example

For example, may want to say that something is

- *possibly* true
- *known* to be true
- *believed* to be true
- . . .

A simple inference using this is

Necessarily, Fred is mortal. Therefore, Fred is mortal.

How can we express this in a logic? First we try a non-modal approach.





# Using FOL?

We could take first order logic, and add a new axiom $\forall x\ nec(x) \rightarrow x$

From this and modus ponens, it looks as though we can get from

$$nec(mortal(fred))$$

to

$$mortal(fred)$$

**BUT** this clashes with our syntax: the two propositions have to be parsed as follows.

$$\overbrace{nec}^{pred}(\overbrace{mortal}^{fn}(\overbrace{fred}^{cst}))$$

$$\underbrace{mortal}_{pred}(\underbrace{fred}_{cst})$$

## Semantics

Also, what about the meaning of the terms here?

In

$$mortal(fred)$$

objects of discourse are *people*; in

$$nec(\ldots)$$

objects of discourse are *propositions* (maybe *formulas*?).

So, though it is possible to build an inference system, it's not clear what the statements in the system *mean*.





## A First-Order Formulation

Extend the syntax by adding for every formula $F$ a new constant ⌜F⌝. Now, for every formula $G$ in the language add the axiom

$$nec(\ulcorner G \urcorner) \rightarrow G$$

For example, we get

$$nec(\ulcorner rich(fred) \urcorner) \rightarrow rich(fred)$$

This is OK for both the syntax, and the semantics;
there are distinct bits of syntax for the *use* and the *mention* of a formula.





## Properties of First-Order version

- Add extra axioms to whatever we already have available.
- Get a first-order theory, so we can use a standard inference engine.
- The syntax is complicated!
- Often we want to make use of the structure of a formula, even when it is mentioned, and we cannot do this in the logic.





## Modal Logic

Instead of adding extra axioms, we add new *logical connectives*.

The standard connectives are

$\Box$ : it is necessary that

$\Diamond$ : it is possible that

We enlarge the syntax definition so that if $F$ is a formula, then so is $\Box F, \Diamond F$.

Many different logics of necessity have been proposed.

# An Inference System

We can give an axiom system by adding three axiom schemes:

| | |
|---|---|
| $\Box(A \to B) \to (\Box A \to \Box B)$ | $Ax1$ |
| $\Box A \to A$ | $Ax2$ |
| $\Box A \to \Box\Box A$ | $Ax3$ |

and a new rule of inference (nec)

$$\text{if } \vdash A \text{ then } \vdash \Box A.$$

We can also *define* $\Diamond$ in terms of $\Box$ by

$$\Diamond A \leftrightarrow \neg\Box\neg A$$

— so $\Diamond A$ is just a shorhand way of writing $\neg\Box\neg A$.





# Derivation

A derivation in modal logic is like one in the predicate calculus with appeal to the new axioms and inference rules.

| | | |
|---|---|---|
| 1 | $p \to (q \to p)$ | *axiom* |
| 2 | $\Box(p \to (q \to p))$ | *necessitation* 1 |
| 3 | $\Box(p \to (q \to p)) \to (\Box p \to \Box(q \to p))$ | *axiomAx*1 |
| 4 | $\Box p \to \Box(q \to p)$ | *modus ponens* 2, 3 |

In the propositional case, this is decidable.





# Necessity

**Necessity** may be understood in several ways.

For example, in a parallel or non-deterministic system, read $\Box F$ as saying that $F$ is true in all branches/in all cases.

Or in game playing, we can read $\Box F$ as saying that $F$ is true, whatever move is made at this point in the game.





# Logic of Knowledge

Let's take $\Box F$ to mean "$F$ is known to be true". How good is our original inference system for this reading?

Are the axioms and inference rules

- *plausible*? (sound)
- *complete*?

In terms of being known, they say:

$$\texttt{known}(a) \to a$$
$$\texttt{known}(a) \to \texttt{known}(\texttt{known}(a))$$
$$\texttt{known}(a \to b) \to (\texttt{known}(a) \to \texttt{known}(b)).$$

Are these OK?

Notice that we *don't* have

$$a \to known(a)$$

What about the necessitation rule:

$$\text{if } \vdash a \text{ then } \vdash known(a)$$

This means that all logical truths are known!

It's hard to find a better formulation here, that allows use of logical inference from knowledge, without assuming that this must be exhaustive.

**Completeness?**

To suggest that the system is not complete, find an intuitively true statement that is not derivable.





# Logics of Belief

Assume that knowledge is *true, justified belief*.

We can build a logic by adding a two place modal connective $bel$ such that is $t$ is a term and $F$ a formula, then $bel(t, F)$ is a formula (intuitively, it expresses that "$t$ believes that $F$").

Now we need appropriate axioms and inference rules.





# Possible axioms

$$bel(x, F) \to bel(x, bel(x, F))$$

Note that we can model inconsistent beliefs in a consistent theory.

$$bel(x, p \to q) \to bel(x, q \to p)$$

We can also express nested beliefs, eg

$$bel(x, bel(y, \neg bel(x, F)))$$





# Introspection

Some rules that treat of reasoning about beliefs in a sequent calculus version are as follows.

**introspect** $$\frac{\mathtt{Forms} \Longrightarrow \mathtt{bel(X, F)}}{\mathtt{Forms} \Longrightarrow \mathtt{bel(X, bel(X, F))}}$$

**belMP** $$\frac{\mathtt{Forms} \Longrightarrow \mathtt{bel(X, F)} \qquad \mathtt{Forms} \Longrightarrow \mathtt{bel(X, F \to G)}}{\mathtt{Forms} \Longrightarrow \mathtt{bel(X, G)}}$$

**belLogic** $$\frac{\Longrightarrow \mathtt{G}}{\mathtt{Forms} \Longrightarrow \mathtt{bel(X, G)}}$$

# Temporal logic

For thinking about agents, we will make some use of *temporal logic*. One approach is to add connectives:

| | |
|---|---|
| $\Box F$ | $F$ is always true |
| $\Diamond F$ | $F$ is eventually true |
| $\bigcirc F$ | $F$ is true at the next time point |
| $F\,\mathbf{U}\,G$ | $F$ is true until $G$ |

We need some rules for reasoning with these modalities.





# Temporal inference

Here is an inference system for temporal logic, using the connectives above.

**Possible Axioms** (schemes for *any* matching formulas)

| | |
|---|---|
| $\Box p \to \Diamond p$ | What will always be, will be. |
| $\Box(p \to q) \to (\Box p \to \Box q)$ | If p always implies q, then if p will always be the case, so will q. |
| $\Diamond p \to \Diamond\Diamond p$ | If it will be the case that p, it will be the case that it will be. |
| $\neg\Diamond p \to \Diamond\neg\Diamond p$ | If it will never be that p, then it will be that it will never be that p. |





# Temporal Logic ctd

**Inference Rules**

- Standard propositional inference
- Necessitation:

  If there is a proof of $p$ (from no assumptions),
  then we can derive a proof of $\Box p$

This is the most basic temporal logic; other machinery is necessary to deal with the other connectives, and issues of discrete vs dense time.





# Summary

For reasoning about

- necessity
- knowledge
- belief
- . . .

**use**

- First-order logic with extra constants, **or**
- Modal logic with new connectives