Knowledge Representation and Engineering
First Assessed Practical

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1. Consider the following sequent presentation of a modal logic of belief. As usual, the left hand side L of a sequent $\mathtt{L} \Longrightarrow \mathtt{R}$ is taken as a *set* of formulas; in this case, there is exactly one formula on the right. Sequents are read in the standard way: a sequent is true if and only if, whenever all the formulas on the left are true, the formula on the right is true. Where there are formulas separated by dots, this means that any number of formulas may be present (perhaps none).

   K is a modal connective that forms a formula when applied to a term, and a formula.

$$\textbf{axiom}\quad \mathtt{F}, \mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \mathtt{F}$$

$$\textbf{andI}\quad \frac{\mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \mathtt{F} \qquad \mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \mathtt{G}}{\mathtt{F}_1 \ldots \mathtt{F}_n \Longrightarrow \mathtt{F} \wedge \mathtt{G}}$$

$$\textbf{andE}\quad \frac{\mathtt{F}, \mathtt{G}, \mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \mathtt{C}}{\mathtt{F} \wedge \mathtt{G}, \mathtt{F}_1, \ldots \mathtt{F}_n, \Longrightarrow \mathtt{C}}$$

$$\textbf{impI}\quad \frac{\mathtt{F}, \mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \mathtt{G}}{\mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \mathtt{F} \rightarrow \mathtt{G}}$$

$$\textbf{impE}\quad \frac{\mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \mathtt{F} \qquad \mathtt{G}, \mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \mathtt{C}}{\mathtt{F} \rightarrow \mathtt{G}, \mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \mathtt{C}}$$

$$\textbf{bel1}\quad \frac{\mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \mathtt{P}}{\mathbf{K}(\mathtt{A}, \mathtt{F}_1), \ldots, \mathbf{K}(\mathtt{A}, \mathtt{F}_n), \mathtt{G}_1, \ldots, \mathtt{G}_m \Longrightarrow \mathbf{K}(\mathtt{A}, \mathtt{P})}$$

$$\textbf{bel2}\quad \frac{\mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \mathbf{K}(\mathtt{A}, \mathtt{P})}{\mathtt{F}_1, \ldots, \mathtt{F}_n \Longrightarrow \mathbf{K}(\mathtt{A}, \mathbf{K}(\mathtt{A}, \mathtt{P}))}$$

   (a) Show that the rule **impE** is sound, given the standard meaning of the implication connective.

   [20%]

   (b) Give a derivation in this system of following sequent (with empty left hand side). You may abbreviate formulas.

$$\begin{aligned} \Longrightarrow \quad (\ & \mathbf{K}(\mathtt{zoe}, \mathbf{K}(\mathtt{yves}, \mathtt{rain} \rightarrow (\mathtt{fire} \rightarrow \mathtt{happiness}))) \\ & \wedge \mathbf{K}(\mathtt{zoe}, \mathbf{K}(\mathtt{yves}, \mathtt{fire})) \\ & \wedge \mathbf{K}(\mathtt{zoe}, \mathbf{K}(\mathtt{yves}, \mathtt{rain}))\ ) \\ & \qquad \rightarrow \mathbf{K}(\mathtt{zoe}, \mathbf{K}(\mathtt{zoe}, \mathbf{K}(\mathtt{yves}, \mathtt{happiness}))) \end{aligned}$$

   [30%]

   (c) Suppose we are given a sequent, and want to find whether or not it has a derivation in this system. We can use use these rules top-down, and use depth-first search to look for a derivation. Explain carefully why provides a *decision procedure* for derivability in this system. (Recall that a decision procedure is an algorithm that correctly returns a value indicating whether the property holds, and always terminates.)

   [25%]

   (d) This is a candidate logic for reasoning about agents' knowledge, where $\mathbf{K}(\mathtt{a}, \mathtt{f})$ is read as "agent a knows that f". Discuss whether the rules **bel1,bel2** are reasonable inference rules when modelling human reasoning. Propose an additional or replacement rule for this notion that you think is plausible in the context, and explain why you find it plausible.

   [25%]

2. Consider the following subset of propositional calculus.

The only formulas allowed as axioms are those of the form

$$F_1 \wedge \cdots \wedge F_n \to G$$

for some atomic propositions $F_1, \ldots, F_n, G$. There may be any number of F_i formulas; in addition a single atomic formula G is allowed is allowed as an axiom.

We assume that the statements we want to prove are always single atomic formulas.

The inference rules are

$$\frac{P \to Q \qquad P}{Q}\textbf{MP} \qquad\qquad \frac{P \qquad Q}{P \wedge Q}\textbf{and}$$

(a) Translate into this language the following argument, and show that the conclusion can be derived using the rules above, by assuming some formulas as axioms and constructing a derivation.

It is dark.
If it is dark, and it is Tuesday, and it is the morning, then there is a tutorial.
If it is dark, it is Tuesday.
If it is Tuesday, it is the morning.
Therefore, there is a tutorial.

[25%]

(b) We can use this system to search for derivations as follows. We try to show a list of atomic goals $G_1, \ldots, G_m$ can all be derived; initially there is one such goal G.

- We iterate the following until there are no goals left (when it returns success) or no axioms to use (when it returns failure):
- – for the first goal G, look for a matching axiom of the form $F_1 \wedge \cdots \wedge F_n \to G$ or G. Delete G from the list of goals, and try to solve the new goal list, $F_1, \ldots, F_n, G_2, \ldots, G_m$ in the first case or $G_2, \ldots, G_m$ in the second case.
  – If this does not solve all the goals, we look for another matching axiom and try again.

Show that this is *not* a complete inference strategy by giving a simple example.

[15%]

(c) Explain why it is that when the algorithm sketched above terminates successfully, we know that the original goal G does have a derivation using the inference rules above.

[20%]

(d) Sketch a decision procedure for this problem, which uses a bottom up method to construct derivations for this system.

[20%]

(e) (Harder) It is also the case that the *inference system* described above is complete, i.e. every logically valid conclusion has a derivation in the system.

Show that this is the case by showing that if all possible rule applications in a bottom-up approach have been applied, and no derivation for the query has been found, then it is possible to construct a model of the axioms (in this case, an assigment of truth values to the propositions making all the axioms true) where the query takes the value *false*.

[20%]

Submission

Submit using the `submit` DICE mechanism; you should submit a single file in text, PDF or PS format. The deadline for submissions is Wednesday 15th February.