Introduction to Theoretical Computer Science

Lecture 2: Regular Languages

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Recall..

DFAs, NFAs and ε-NFAs all recognise the same class of languages, called the *regular languages*. They are equal in expressive power, although some representations (NFAs) are more compact than others (DFAs).

Closure Properties

Definition

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That is, if we have two regular languages L_1 and L_2, is $L_1 \cup L_2$ also regular?
Exercise: Prove this.

Closure Properties

Definition

The *sequential composition* of two languages L_1 and L_2, written $L_1 L_2$, is the language of strings that consist of a string in L_1 followed by a string in L_2.

$$L_1 L_2 = \{vw \mid v \in L_1, w \in L_2\}$$

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Exercise: Prove this.

Closure Properties

Notation

Similarly to arithmetic, define L^0 as $\{\varepsilon\}$ and $L^{n+1} = LL^n$.

Definition

The *Kleene closure* of a language L, written L^*, is the language of strings that consist wholly of zero or more strings in L.

$$L^* = \bigcup_{i \in \mathbb{N}} L^i$$

(n.b: in computer science, $0 \in \mathbb{N}$)

Are the regular languages *closed* under Kleene closure?

Exercise: Prove this.

Regular Expressions

Regular expressions are an algebraic notation for regular languages. Many of you will have already used (some variant of) regular expressions in your text editors.

Syntax	Semantics
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$R*$	$[\![R*]\!] = [\![R]\!]^*$

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Questions

- How do we write "at least one `0`"? What about "at least one `0` and at least one `1`?"

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- How do we write $R?$, the *optional* R, using existing operators?

Regular Expressions vs Finite Automata

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- **RE** $\rightarrow$ **DFA** – apply the constructions used in our closure proofs, then the subset construction.
- **DFA** $\rightarrow$ **RE** – convert to a *generalised NFA*, then reduce to a single transition.

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- **DFA** $\rightarrow$ **RE** – convert to a *generalised NFA*, then reduce to a single transition.

A note

The DFAs we get from our **RE** $\rightarrow$ **DFA** translation are not very space-efficient. Most implementations use more advanced techniques to minimise the DFA.

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- The transition relation is *full*, except that the initial state has no incoming transitions, and the final state has no outgoing transitions.

(n.b: transitions can be labelled with $\emptyset$)

What do we need to do to convert a DFA to a GNFA?

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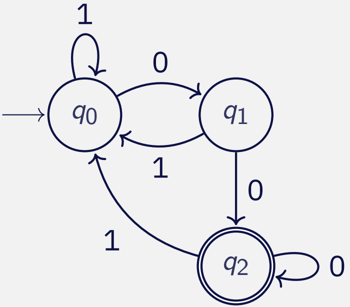
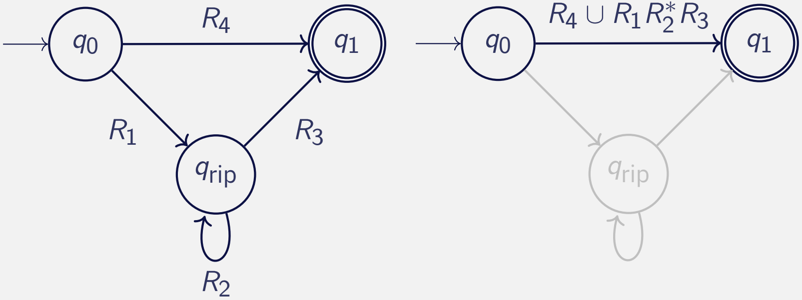


GNFA to RE

We will eliminate each of the inner states of the GNFA one by one. When all of them are gone, only the initial and final state will remain, with one transition between them. The label on this transition will be our regular expression.



Exercise: Let's reduce our example to a single RE.