Introduction to Theoretical Computer Science

Lecture 18: Denotational Semantics

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Semester 1, 2023/2024

Semantics

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Why learn this?

We can't prove anything about a computer program without first giving it a semantics.

Semantics

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In this lecture

We focus mostly on denotational semantics as MCS's treatment is very informal and no other course touches it.

Denotational Semantics

At its heart, it's quite simple:

$$[\![\cdot]\!] : \text{Program} \to \text{Semantics}$$

More specifically, we define a function $[\![\cdot]\!]$ which maps *syntax* into (mathematical) *models*.

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Desideratum

We want this semantic function to be *compositional*: The semantics of a compound expression should be made from the semantics of its components.

Robot Example

Example (A Toy Language)

A robot moves along a grid according to a sequence of commands `move` (forward 1 unit) and `turn` (90 degrees counter-clockwise), separated by semicolons, with the command sequence terminated by the keyword `stop`:

$$\mathcal{R} ::= \texttt{move};\ \mathcal{R} \mid \texttt{turn};\ \mathcal{R} \mid \texttt{stop}$$

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$$\begin{aligned}
[\![\cdot]\!]^{\mathcal{R}} &: \mathcal{R} \to \mathbb{Z}^2 \\
[\![\texttt{turn};\ r]\!]^{\mathcal{R}} &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} [\![r]\!]^{\mathcal{R}} \\
[\![\texttt{move};\ r]\!]^{\mathcal{R}} &= [\![r]\!]^{\mathcal{R}} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
[\![\texttt{stop}]\!]^{\mathcal{R}} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{aligned}$$

Arithmetic Expressions

$$[\![\cdot]\!]^{\mathcal{E}} : \mathcal{E} \to$$

$$
\begin{aligned}
[\![n]\!]^{\mathcal{E}} &= \\
[\![x]\!]^{\mathcal{E}} &= \\
[\![e_1 \mathbin{\textbf{+}} e_2]\!]^{\mathcal{E}} &= \\
[\![e_1 \mathbin{\textbf{*}} e_2]\!]^{\mathcal{E}} &= \\
[\![\textbf{let } x = e_1 \textbf{ in } e_2]\!]^{\mathcal{E}} &=
\end{aligned}
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$$[\![\cdot]\!]^{\mathcal{E}} : \mathcal{E} \to \mathbb{Z}$$

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[\![x]\!]^{\mathcal{E}} &= \\
[\![e_1 + e_2]\!]^{\mathcal{E}} &= \\
[\![e_1 * e_2]\!]^{\mathcal{E}} &= \\
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$$[\![\cdot]\!]^{\mathcal{E}} : \mathcal{E} \to \Sigma \to \mathbb{Z}$$

Our denotation for arithmetic expressions is functions from *states* (mapping from variables to their values) to values.

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[\![e_1 \mathbin{\textbf{*}} e_2]\!]^{\mathcal{E}} &= \lambda\sigma.\ [\![e_1]\!]^{\mathcal{E}}\sigma \times [\![e_2]\!]^{\mathcal{E}}\sigma \\
[\![\textbf{let}\ x = e_1\ \textbf{in}\ e_2]\!]^{\mathcal{E}} &= \lambda\sigma.\ [\![e_2]\!]^{\mathcal{E}}\left(\sigma[x := [\![e_1]\!]^{\mathcal{E}}\sigma]\right)
\end{aligned}$$

Where $\sigma[x := n]$ is a new state just like σ except the variable x now maps to n.

Note: From this point onwards I'll assume all standard arithmetic expressions are in $\mathcal{E}$

Boolean Expressions

$$[\![\cdot]\!]^{\mathcal{B}} : \mathcal{B} \to$$

$$\begin{aligned}
[\![e_1 \texttt{ == } e_2]\!]^{\mathcal{B}} &= \\
[\![e_1 \texttt{ <= } e_2]\!]^{\mathcal{B}} &= \\
[\![e_1 \texttt{ \&\& } e_2]\!]^{\mathcal{B}} &= \\
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$$[\![\cdot]\!]^{\mathcal{B}} : \mathcal{B} \rightarrow \mathcal{P}(\Sigma)$$

$$\begin{aligned}
[\![e_1 \texttt{ == } e_2]\!]^{\mathcal{B}} &= \{\sigma \mid [\![e_1]\!]^{\mathcal{E}}\sigma = [\![e_2]\!]^{\mathcal{E}}\sigma\} \\
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[\![\texttt{! } e_1]\!]^{\mathcal{B}} &= \Sigma \setminus [\![e_1]\!]^{\mathcal{B}}
\end{aligned}$$

Note: C notation is used here to distinguish syntax from semantics, but from this point onwards I'll assume all standard boolean expressions are in $\mathcal{B}$

Imperative Programs

We are going to give semantics to non-deterministic imperative programs. Because of non-determinism, our models are relations not functions:

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Assignment statement

An *assignment* $x := e$ simply assigns the value of the expression e to the variable x:

$$[\![x := e]\!] = \left\{(\sigma_i, \sigma_f) \mid \sigma_f = \sigma_i \left[x \mapsto [\![e]\!]^{\mathcal{E}}(\sigma_i)\right]\right\}$$

More Statements

Sequencing

The semicolon, or *sequential composition* operator, is the operator that lets us first run P, and then run Q.

$$\llbracket P; Q \rrbracket = \llbracket P \rrbracket \fatsemi \llbracket Q \rrbracket$$

where $\fatsemi$ is forward-composition of relations:

$$X \fatsemi Y = \{(\sigma_i, \sigma_f) \mid \exists \sigma_m.\ (\sigma_i, \sigma_m) \in X \wedge (\sigma_m, \sigma_f) \in Y\}$$

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Example (Swap)

$$(\{a \mapsto 4, b \mapsto 8, \ldots\}, \{a \mapsto 8, b \mapsto 4, \ldots\}) \in [\![x := a; a := b; b := x]\!]$$

More Statements

Choice and Guards

An *a nondeterministic choice* $P + Q$ means that all observations of P and all observations of Q are possible:

$$[\![P + Q]\!] = [\![P]\!] \cup [\![Q]\!]$$

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A boolean expression *guard* φ (in $\mathcal{B}$) doesn't change the state, but only those observations that satisfy φ succeed:

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Using these ingredients, we can recover **if**-statements:

$$\textbf{if } \varphi \textbf{ then } P \textbf{ else } Q \textbf{ fi} \simeq (\varphi; P) + (\neg\varphi; Q)$$

Loops

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Star

The *Kleene star* $P^\star$ is the operator that runs loop body P for a nondeterministic amount of times. The semantics are the smallest solution to this recursive equation:

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We will show that this is the same as:

$$[\![P^\star]\!] = \bigcup_{i \in \mathbb{N}_0} [\![P]\!]^i$$

Where superscripting is self-composition:

$$\begin{aligned} R^0 &= I \\ R^{n+1} &= R \mathbin{;} R^n \end{aligned}$$

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We can recover **while** loops: **while** g **do** P **od** $\simeq (g; P)^\star; \neg g$

Great Scott!

Rewriting our equation slightly:

$$[\![P^\star]\!] = f([\![P^\star]\!]) \text{ where } f(X) = I \cup [\![P]\!] \mathbin{;} X$$

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1. *Pointed*: it has a least element $\bot$ which approximates everything.
2. *ω-chain-complete*: For every countable ascending sequence $f_0 \sqsubseteq f_1 \sqsubseteq f_2 \ldots$ we have a least upper bound, written $\sup f$ or $\bigsqcup_{n \in \mathbb{N}} f_n$.

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- The least element $\perp = \emptyset$
- The least upper bound of a chain $f_0 \subseteq f_1 \subseteq f2 \ldots$ is just $\bigcup_{i \in \mathbb{N}} f_i$

Climbing Chains

Recalling our semantics for the star operator, we want to show that the least fixed point of a function f on our cpo is the least upper bound of the *ascending Kleene chain*:

$$\bot \sqsubseteq f(\bot) \sqsubseteq f(f(\bot)) \sqsubseteq f^3(\bot) \sqsubseteq f^4(\bot) \sqsubseteq \cdots$$

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But!

This chain doesn't exist for some f! Consider this f on the flat domain $(\mathbb{N}_\bot, \sqsubseteq)$:

$$f(x) = \begin{cases} 1 & \text{if } x = \bot \\ \bot & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases}$$

Climbing Chains

Recalling our semantics for the star operator, we want to show that the least fixed point of a function f on our cpo is the least upper bound of the *ascending Kleene chain*:

$$\bot \sqsubseteq f(\bot) \sqsubseteq f(f(\bot)) \sqsubseteq f^3(\bot) \sqsubseteq f^4(\bot) \sqsubseteq \cdots$$

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Requiring that f is *monotone* fixes this problem, i.e. $a \leq b \implies f(a) \leq f(b)$. Why?

Monotone isn't enough

Consider this function f defined over a cpo $(\mathbb{R} \cup \{-\infty, \infty\}, \leq)$:

$$f(x) = \begin{cases} \tan^{-1} x & \text{if } x < 0 \\ 1 & \text{otherwise} \end{cases}$$

Note that this function is not continuous at 0.

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Oh no

It has a fixed point of 1, but the chain approaches 0:

$$\begin{array}{lcl} f(-\infty) & = & -\frac{\pi}{2} \\ f(-\frac{\pi}{2}) & = & -1 \\ f(-1) & \approx & -0.78 \end{array}$$

But $f(0) = 1$ — the least upper bound of the ascending Kleene chain is not the same as the least fixed point!

Continuity

Definition

In a cpo $(S, \sqsubseteq)$, a function $f : S \to S$ is *(Scott)-continuous* if, for every chain $x_0 \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq \ldots$, f preserves the least upper bound operator:

$$\bigsqcup_{n \in \mathbb{N}} f(x_n) = f\Big(\bigsqcup_{n \in \mathbb{N}} x_n\Big)$$

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Requiring Scott-continuity instead of just monotonicity gives us the Kleene fixed point theorem...

The Kleene fixed point theorem

Theorem

Let $(S, \sqsubseteq)$ be a cpo and $f : S \to S$ be a Scott-continuous function. Then the lub of the Kleene ascending chain $\bigsqcup_{n \in \mathbb{N}} f^n(\bot)$ is the least fixed point of f.

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Proof it is a fixed point:

$$f(\bigsqcup_{n \in \mathbb{N}} f^n(\bot)) = \bigsqcup_{n \in \mathbb{N}} f(f^n(\bot)) \qquad \text{(continuity)}$$

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Bringing it back to semantics

For our programming language, our cpo is $(\mathcal{P}(\Sigma \times \Sigma), \subseteq)$:

- The least element $\bot = \emptyset$
- The least upper bound of a chain $f_0 \subseteq f_1 \subseteq f2 \ldots$ is just $\bigcup_{i \in \mathbb{N}} f_i$

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Thus, we know from the fixed point theorem that least solutions to our recursive equations always exist and they can be found by iteratively applying the function until we find a fixed point.

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Key idea

Add a special value, confusingly also written $\bot$, which represents non-terminating computations. Our models would now be $\mathcal{P}(\Sigma \times \Sigma_\bot)$ where $\Sigma_\bot$ is either a state or the special “loop forever” value.

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It's quite tricky to define this ordering such that it is a cpo and such that our language operations are still continuous.

Further reading

Plotkin resolved this with his Powerdomain construction, which gives a general treatment of non-determinism such that any cpo can be lifted to a non-deterministic context.

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- *Adequacy* If $(\sigma, \sigma') \in [\![P]\!]$ then $(\sigma, P) \Downarrow \sigma'$
- *Full Abstraction* $[\![P]\!] = [\![Q]\!]$ iff for all contexts C and states σ and σ', $(\sigma, C[P]) \Downarrow \sigma' \Leftrightarrow (\sigma, C[Q]) \Downarrow \sigma'$

The first two are common. The last one is hard.

More on denotations

This is just the tip of the iceberg in Denotational Semantics.

- Effectful programs use Kleisli categories (monads) for their domain
- Categorical semantics which use structures from category theory for denotations.
- Game semantics which use games as denotations.
- Probabilistic powerdomains and quasi-Borel spaces for probablistic programs.
- Concurrency semantics using traces, transition systems, event structures, Petri nets and so on. MCS

Farewell

Best of luck with your exams and the rest of your life! Please feel free to reach out if you're interested in learning more theory.