Introduction to Theoretical Computer Science

Lecture 13: The Polynomial Hierarchy, Alternation and PSPACE

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The Class **CoNP**

trick **Question**

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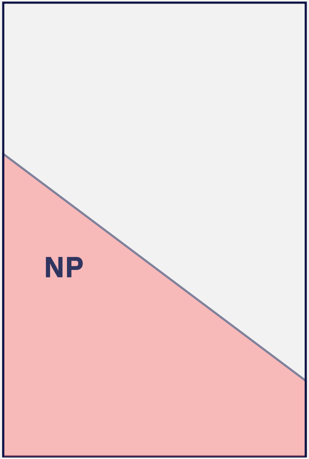

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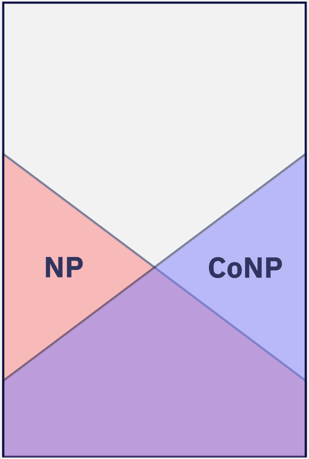

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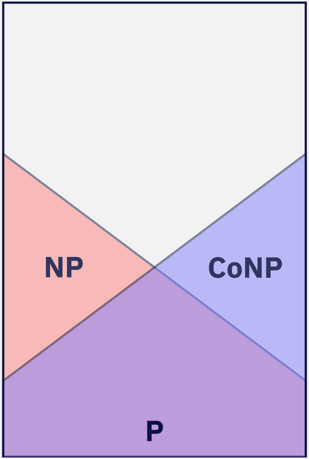

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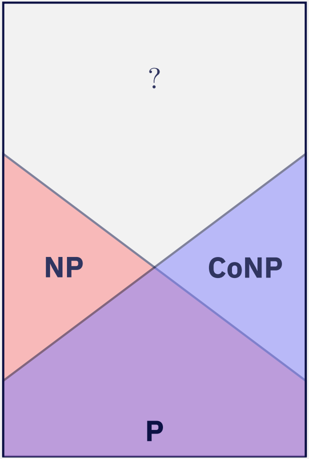

Sigmas

We shall introduce notation to describe polynomial problems.

Sigma

The set $\Sigma_1^{\mathbf{P}}$ describes all problems that can be phrased as $\{y \mid \exists^{\mathbf{P}} x \in \mathbb{N}.\ R(x, y)\}$, where R is a **P**-decidable predicate and $\exists^{\mathbf{P}} x \ldots$ indicates that x is of size polynomial in the size of y.

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Certificates

We can say that x is a *certificate* showing which "guesses" can made by our NRM giving an accepting run.

So, **NP** $= \Sigma_1^{\mathbf{P}}$.

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$$\begin{aligned}
\overline{\Sigma_1^{\mathbf{P}}} &= \overline{\{x \mid \exists^{\mathbf{P}} y.\ R(x, y)\}} \\
&= \{x \mid \neg\exists^{\mathbf{P}} y.\ R(x, y)\} \\
&= \{x \mid \forall^{\mathbf{P}} y.\ \neg R(x, y)\} \\
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As $\Sigma_1^{\mathbf{P}}$ is **NP**, $\Pi_1^{\mathbf{P}}$ is **CoNP**.

Deltas

For reasons that are unknown to me, some books have:

Delta

The set Δ_1^{P} describes the intersection of Σ_1^{P} and Π_1^{P}.

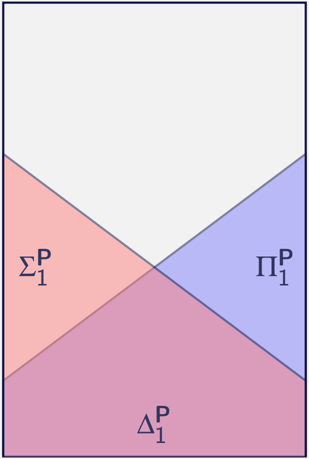
While others have

Delta

The set Δ_1^{P} describes the set P

From our characterisations of Σ_1^{P} and Π_1^{P}, we have that $\Delta_1^{\mathsf{P}} \supseteq \mathbf{P}$, but we don't know if these definitions are equal.

Relabeling



Moving Higher

Definitions

- Σ_2^{P} is all problems of form $\{x \mid \exists^{\mathsf{P}} y . \forall^{\mathsf{P}} z.\ R(x, y, z)\}$.
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Note that $\Sigma_1^{\mathsf{P}}, \Pi_1^{\mathsf{P}}, \Delta_1^{\mathsf{P}}$ are all $\subseteq \Delta_2^{\mathsf{P}}$ (and therefore $\subseteq \Sigma_2^{\mathsf{P}}$ and $\subseteq \Pi_2^{\mathsf{P}}$). **Why?**

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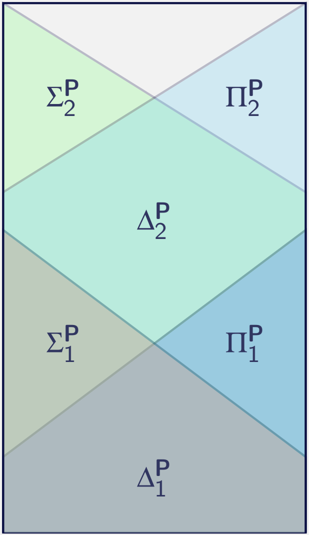
(our R can simply "ignore" one of the parameters)

The Polynomial Hierarchy



An equivalent characterisation

We can define in terms of oracles:

- Δ_2^P is all problems that are decidable in polynomial time by some deterministic TM/RM with an $\mathcal{O}(1)$ oracle for some complete problem in Σ_1^P, i.e. it is **P** with an $\mathcal{O}(1)$ oracle for **NP**.
- Σ_2^P allows the TM/RM to be nondeterministic, i.e. it is **NP** with an $\mathcal{O}(1)$ oracle for **NP**.
- Π_2^P is **CoNP** with an oracle for **NP**.

Building up

In general, for any $n > 1$:

- $\Delta_n^{\mathbf{P}}$ is all problems that are decidable by some deterministic, polynomially bounded TM/RM with an $\mathcal{O}(1)$ oracle for some problem $\in \Sigma_{n-1}^{\mathbf{P}}$.

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- Π_n^{P} are all problems decidable by some *co-nondeterministic*, polynomially bounded TM/RM with an $\mathcal{O}(1)$ oracle for some problem $\in \Sigma_{n-1}^{\mathsf{P}}$.

Co-nondeterminism

Could also be called *demonic* nondeterminism. Like our normal (angelic) nondeterminism but only accepts if *all* paths accept.

Alternation

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Equivalently $\Sigma_n^{\mathbf{P}}$ are all problems that can be phrased as some alternation of (**P**-bounded) quantifiers, starting with $\exists^{\mathbf{P}}$:

$$\{w \mid \exists^{\mathbf{P}} x_1.\forall^{\mathbf{P}} x_2.\exists^{\mathbf{P}} x_3.\forall^{\mathbf{P}} x_4.\ldots x_n.\ R(w, x_1, \ldots, x_n)\}$$

$\Pi_n^{\mathbf{P}}$ starts instead with $\forall^{\mathbf{P}}$:

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Alternating Register Machines

Consider NRMs where instead of just a MAYBE instruction we have a $\text{MAYBE}^{\forall}$ instruction and a $\text{MAYBE}^{\exists}$ instruction.

- $\text{MAYBE}^{\exists}$ is a nondeterministic choice where we accept if one branch accepts.
- $\text{MAYBE}^{\forall}$ is a nondeterministic choice where we accept only when both branches accept.

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Alternating Turing Machines are defined by labelling states with either $\forall$ or $\exists$.

Alternating Machines and the Polynomial Hierarchy

- The class $\Sigma_n^{\mathbf{P}}$ could equivalently be defined as the class of problems decided in polynomial time by an alternating machine that initially uses $\exists$-nondeterminism, and every path in the machine swaps quantifiers (i.e. to $\forall$ or back to $\exists$) at most $n - 1$ times.

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AP is known to be equal to **PSPACE** (more on this in a moment)

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PSPACE $\supseteq$ **P** ? **PSPACE** $\supseteq$ **NP** ? **PSPACE** $\subseteq$ **EXPTIME**?

Conclusions

This concludes our study of complexity theory. Next week, we start on a new (or rather, very old) model of computation that is the foundation for modern studies of programming languages and their semantics, the λ-calculus.