Introduction to Theoretical Computer Science

Lecture 10 [bonus]: Games

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- A logical game is *finite* if there is an n such that all plays are determined to be in $\mathcal{W}_\exists$ or $\mathcal{W}_\forall$ based on a finite prefix of length n.

Winning Strategies

A logical game is *determined* if one or the other players have a *winning strategy*.

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$\exists$-winning strategy: we have a proof of φ.

$\forall$-winning strategy: we have a counterexample to φ.

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Every well-founded game is determined.

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- If $\exists$ moves, she must have a move that does not put $\forall$ into a winning strategy, or otherwise the previous position would have a $\forall$-winning strategy.
- Thus, inductively, the entire run will never put $\forall$ in a winning position. Thus, $\exists$ has won.

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A formula φ holds iff $\exists$ has a winning strategy for $G[\varphi]$.

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$$[\mathbf{lfp}_{R(\vec{x})}.\,\varphi](\vec{y}) \quad \equiv \quad \varphi\left[{}^{\vec{y}}/{}_{\vec{x}}\right]\left[{}^{[\mathbf{lfp}_{R(\vec{x})}.\,\varphi](\vec{z})}/{}_{R(\vec{z})}\right]$$

Example (Solitaire Games)

Given a graph consisting of a connectness predicate $E(a, b)$, the cycle-finding game can be stated as:

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Why is this a solitaire game?

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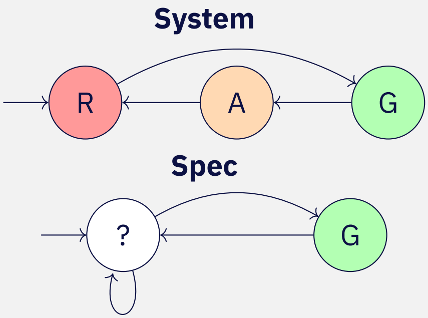
- **D** picks a “matching” element of B.
- **S** wins if he picks an element that **D** cannot match.
- **D** wins if she can continue matching **S**’s moves forever.

Simulation Games

Consider a traffic light system and its specification:



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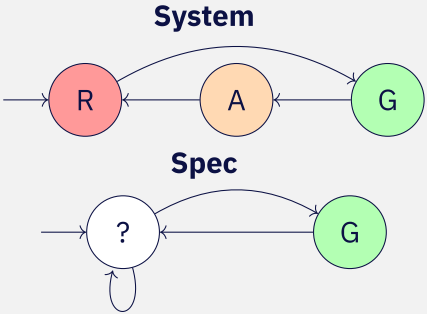

Abstraction

Showing that the system meets the spec requires a *simulation relation*: a winning strategy for a back and forth game where **S** picks system moves and **D** picks matching spec moves.

Simulation Relations

Definition

A *simulation* of an automaton C by an automaton A is defined as a relation $\mathcal{S} \subseteq Q_C \times Q_A$ which satisfies:

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The automaton A is an *abstraction* of the concrete automaton C iff a A simulates C. This is sometimes written $A \sqsubseteq C$.

Simulation relations are the foundation of abstraction – a key technique in formal modelling and verification.

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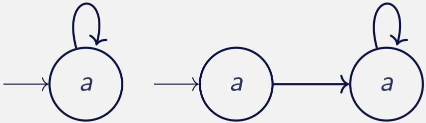

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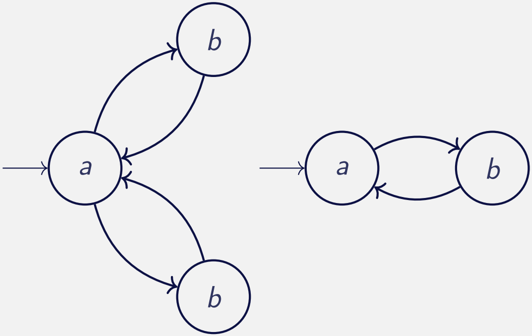

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Let's find bisimulations for the previous examples.

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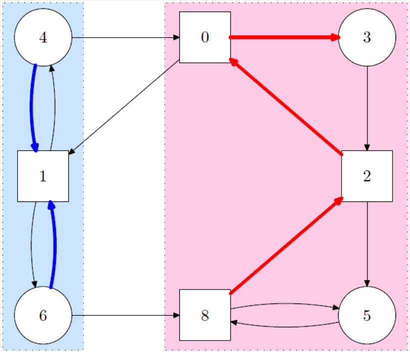
- If **S** can find a move that **D** cannot match, **S** wins.
- **D** wins if it can match all moves selected by **S**.

Parity Games

Definition

A parity game is played between two players on a directed graph. Player 0 chooses moves from circular nodes and Player 1 chooses for square nodes. Player n wins an infinite play if the highest number infinitely visited in the play $\equiv n \; [\mathbf{mod}\ 2]$.



Parity Games

- Parity games can be used to give a model-checking algorithm for a type of logic called *modal μ-calculus*, commonly used to express properties of systems.
- Validity and satisfiability for many other modal logics is reducible to parity game solving.
- Parity games are history-free determined.
- Zielonka gives an algorithm for solving parity games.
- **Open question**: Can parity games be solved in polynomial time?