Information Theory

http://www.inf.ed.ac.uk/teaching/courses/it/

Week 2
Information and Entropy

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Numerics: $\log \sum_i \exp(x_i)$

$\binom{N}{k}$ blows up for large N, k; we evaluate $l_{N,k} = \ln \binom{N}{k}$

Common problem: want to find a sum, like $\sum_{k=0}^{t} \binom{N}{k}$

Actually we want its log:

$$\ln \sum_{k=0}^{t} \exp(l_{N,k}) = l_{\max} + \ln \sum_{k=0}^{t} \exp(l_{N,k} - l_{\max})$$

To make it work, set $l_{\max} = \max_k \, l_{N,k}$. `logsumexp` functions are frequently used

Distribution over blocks

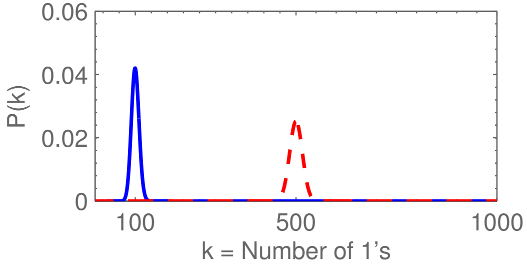
total number of bits: N ($=1000$ in examples here)
probability of a 1: $p = P(x_i{=}1)$
number of 1's: $k = \sum_i x_i$

Every block is improbable!

$P(\mathbf{x}) = p^k (1-p)^{N-k}$, (at most $(1{-}p)^N \approx 10^{-45}$ for $p{=}0.1$)

How many 1's will we see?

$P(k) = \binom{N}{k} p^k (1-p)^{N-k}$

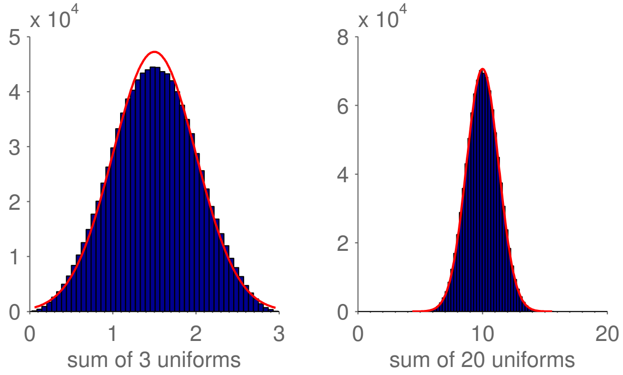


Solid: $p{=}0.1$

Dashed: $p{=}0.5$

Central Limit theorem

The sum or mean of independent variables with bounded mean and variance tends to a Gaussian (normal) distribution.



```
N=1e6;  hist(sum(rand(3,N),1));  hist(sum(rand(20,N),1));
```

There are a few forms of the Central Limit Theorem (CLT), we are just noting a vague statement as we won't make extensive use of it.

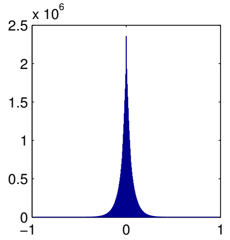
CLT behaviour can occur unreasonably quickly when the assumptions hold. Some old random-number libraries used to use the following method for generating a sample from a unit-variance, zero-mean Gaussian: a) generate 12 samples uniformly between zero and one; b) add them up and subtract 6. It isn't that far off!

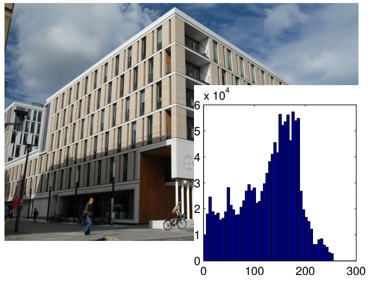
Data from a natural source will usually *not* be Gaussian. The next slide gives examples. Reasons: extreme outliers often occur; there may be lots of strongly dependent variables underlying the data; there may be mixtures of small numbers of effects with very different means or variances.

An example random variable with unbounded mean is given by the payout of the game in the *St. Petersburg Paradox*. A fair coin is tossed repeatedly until it comes up tails. The game pays out $2^{\#\text{heads}}$ pounds. How much would you pay to play? The 'expected' payout is infinite: $1/2 \times 1 + 1/4 \times 2 + 1/8 \times 4 + 1/16 \times 8 + \ldots = 1/2 + 1/2 + 1/2 + 1/2 + \ldots$

Gaussians are not the only fruit

```
xx = importdata('Holst_-_Mars.wav');
hist(double(xx(:)), 400);
```





```
xx = importdata('forum.jpg');
hist(xx(:), 50);
```

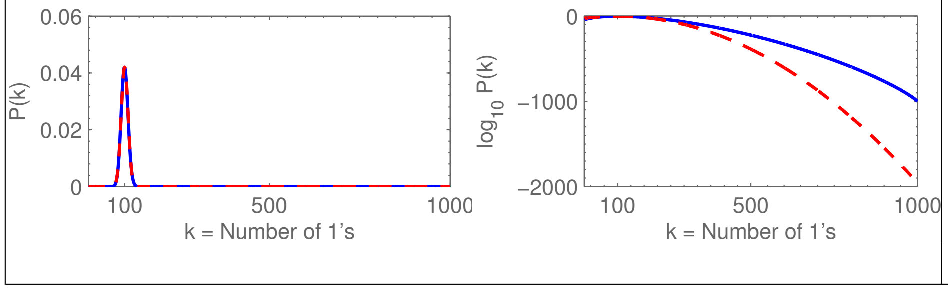
How many 1's will we see?

How many 1's will we see? $P(k) = \binom{N}{k} p^k (1-p)^{N-k}$

Gaussian fit (dashed lines):

$$P(k) \approx \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(k-\mu)^2\right), \quad \mu = Np, \quad \sigma^2 = Np(1-p)$$

(Binomial mean and variance, MacKay p1)



The log-probability plot on the previous slide illustrates how one must be careful with the Central Limit Theorem. Even though the assumptions hold, convergence of the tails is very slow. (The theory gives only "*convergence in distribution*" which makes weak statements out there.) While k, the number of ones, closely follows a Gaussian near the mean, we can't use the Gaussian to make precise statements about the tails.

All that we will use for now is that the mass in the tails further out than a few standard deviations (a few σ) will be small. This is correct, we just can't guarantee that the probability will be quite as small as if the whole distribution actually were Gaussian.

Chebyshev's inequality (MacKay p82, Wikipedia, ...) tells us that:

$$P(|k-\mu| \geq m\sigma) \leq \frac{1}{m^2},$$

a loose bound which will be good enough for what follows.

The fact that as $N \to \infty$ all of the probability mass becomes close to the mean is referred to as the *law of large numbers*.

Encode the *typical set*

Index almost every block we'll see, with $k_{\min} \le k \le k_{\max}$:

$$k_{\min} = \mu - m\sigma$$
$$k_{\max} = \mu + m\sigma$$

$m\!=\!4$ ought to do it (but set much larger to satisfy Chebyshev's if you like)

How many different blocks are in our set?

Probabilities:

— Most probable block: $P_{\max} = p^{k_{\min}}(1-p)^{N-k_{\min}}$
— Least probable block: $P_{\min} = p^{k_{\max}}(1-p)^{N-k_{\max}}$

Probabilities add up to one $\Rightarrow$ **Bound on set size I:**

$$I < \frac{1}{P_{\min}} \quad \Rightarrow \quad \log I < -k_{\max}\log p - (N\!-\!k_{\max})\log(1\!-\!p)$$

Asymptotic possibility

Encoding the set will take $\left(\frac{1}{N}\log_2 I\right)$ bits/symbol

$$\frac{1}{N}\log I < -\frac{1}{N}(\mu\!+\!m\sigma)\log p - \frac{1}{N}(N\!-\!\mu\!-\!m\sigma)\log(1\!-\!p)$$
$$= -\left(p\!+\!m\sqrt{\tfrac{p(1-p)}{N}}\right)\log p - \left(1\!-\!p\!-\!m\sqrt{\tfrac{p(1-p)}{N}}\right)\log(1\!-\!p)$$

As $N\!\rightarrow\!\infty$ for sets of *any* width m:

$$\tfrac{1}{N}\log I \;<\; H_2(p) = -p\log p - (1\!-\!p)\log(1\!-\!p) \quad \approx 0.47 \text{ bits}$$

$(p\!=\!0.1)$

Large sparse blocks can be compressed to NH_2 bits.

Asymptotic impossibility

Large blocks almost always fall in our *typical set*, $T_{N,m}$
Idea: try indexing a set S with $N(H_2\!-\!\epsilon)$ bits

$$\begin{aligned} P(\mathbf{x}\in S) \;&=\; P(\mathbf{x}\in S\cap T_{N,m}) + P(\mathbf{x}\in S\cap \overline{T_{N,m}})) \\ &\le\; 2^{N(H_2-\epsilon)}P_{\max} \;+\; \text{“tail probability”} \end{aligned}$$

$\left[\log P_{\max} = -N\big(H_2 + \mathcal{O}(\tfrac{1}{\sqrt{N}})\big),\right.$ derivation similar to last slide $\left.\right]$

$$P(\mathbf{x}\in S) \;\le\; 2^{-N(\epsilon+\mathcal{O}(1/\sqrt{N}))} \;+\; \text{“tail probability”}$$

The probability of landing in any set indexed by fewer than H_2 bits/symbol becomes tiny as $N\rightarrow\infty$

A weighing problem

Find 1 odd ball out of 12

You have a two-pan balance with three outputs:
“left-pan heavier”, “right-pan heavier”, or “pans equal”

How many weighings do you need to find the odd ball *and* decide whether it is heavier or lighter?

Unclear? See p66 of MacKay's book, but do not look at his answer until you have had a serious attempt to solve it.

Are you sure your answer is right? Can you prove it?
Can you prove it without an extensive search of the solution space?

Weighing problem: bounds

Find 1 odd ball out of 12 with a two-pan balance

There are 24 hypothesis:

ball 1 heavier, ball 1 lighter, ball 2 heavier, . . .

For K weighings, there are at most 3^K outcomes:

(left, balance, right), (right, right, left), . . .

$3^2 = 9 \;\Rightarrow\;$ 2 weighings not enough

$3^3 = 27 \;\Rightarrow\;$ 3 weighings *might* be enough

Weighing problem: strategy

Find 1 odd ball out of 12 with a two-pan balance

Probability of an outcome is: $\frac{\text{\# hypotheses compatible with outcome}}{\text{\# hypotheses}}$

Experiment	Left	Right	Balance
1 vs. 1	$2/24$	$2/24$	$20/24$
2 vs. 2	$4/24$	$4/24$	$16/24$
3 vs. 3	$6/24$	$6/24$	$12/24$
4 vs. 4	$8/24$	$8/24$	$8/24$
5 vs. 5	$10/24$	$10/24$	$4/24$
6 vs. 6	$12/24$	$12/24$	$0/24$

Weighing problem: strategy

8 hypotheses remain. Find a second weighing where:

3 hypotheses ⇒ left pan down
3 hypotheses ⇒ right pan down
2 hypotheses ⇒ balance

It turns out we can always identify one hypothesis with a third weighing (p69 MacKay for details)

Intuition: outcomes with even probability distributions seem *informative* — useful to identify the correct hypothesis

Sorting (review?)

How much does it cost to sort n items?

There are 2^C outcomes of C binary comparisons

There are $n!$ orderings of the items

To pick out the correct ordering must have:

$$C \log 2 \geq \log n! \quad\Rightarrow\quad C \geq \mathcal{O}(n \log n) \qquad \text{(Stirling's series)}$$

Radix sort is "$\mathcal{O}(n)$", gets more information from the items

Measuring information

As we read a file, or do experiments, we get **information**

Very probable outcomes are not informative:

⇒ Information is zero if $P(x)\!=\!1$

⇒ Information increases with $1/P(x)$

Information of two independent outcomes add

⇒ $f\big(\frac{1}{P(x)P(y)}\big) = f\big(\frac{1}{P(x)}\big) + f\big(\frac{1}{P(y)}\big)$

Shannon information content: $h(x) = \log \frac{1}{P(x)} = -\log P(x)$

The base of the logarithm scales the information content:

base 2: bits

base e: nats

base 10: bans (used at Bletchley park: MacKay, p265)

$\log \frac{1}{P}$ **is the only natural measure of information based on probability alone** (matching certain assumptions)

Assume: $f(ab) = f(a) + f(b)$; $f(1) = 0$; f smoothly increases

$$f(a(1+\epsilon)) = f(a) + f(1+\epsilon)$$

Take limit $\epsilon \rightarrow 0$ on both sides:

$$f(a) + a\epsilon f'(a) = f(a) + \cancelto{0}{f(1)} + \epsilon f'(1)$$

$$\Rightarrow f'(a) = f'(1)\frac{1}{a}$$

$$\int_1^x f'(a)\;\mathrm{d}a = f'(1)\int_1^x \frac{1}{a}\;\mathrm{d}a$$

$$f(x) = f'(1)\ln x$$

Define $b = e^{1/f'(1)}$, which must be >1 as f is increasing.

$$f(x) = \log_b x$$

We can choose to measure information in any base ($>$1), as the base is not determined by our assumptions.

Foundations of probability *(very much an aside)*

The main step justifying information resulted from $P(a,b) = P(a)\,P(b)$ for independent events. Where did *that* come from?

There are various formulations of probability. Kolmogorov provided a measure-theoretic formalization for frequencies of events.

Cox (1946) provided a very readable rationalization for using the standard rules of probability to express beliefs and to incorporate knowledge: `http://dx.doi.org/10.1119/1.1990764`

There's some (I believe misguided) arguing about the details. A sensible response to some of these has been given by Van Horn (2003) `http://dx.doi.org/10.1016/S0888-613X(03)00051-3`

Ultimately for both information and probability, the main justification for using them is that they have proven to be hugely useful. While one can argue forever about choices of axioms, I don't believe that there are other compelling formalisms to be had for dealing with uncertainty and information.

Information content vs. storage

A 'bit' is a symbol that takes on two values.
The 'bit' is also a unit of information content.

Numbers in 0–63, e.g. $47\!=\!101111$, need $\log_2 64 = 6$ bits

If numbers 0–63 are equally probable, being told the number has information content $-\log\frac{1}{64} = 6$ bits

The binary digits are the answers to six questions:

1: is $x \geq 32$?
2: is $x \bmod 32 \geq 16$?
3: is $x \bmod 16 \geq 8$?
4: is $x \bmod 8 \geq 4$?
5: is $x \bmod 4 \geq 2$?
6: is $x \bmod 2 = 1$?

Each question has information content $-\log\frac{1}{2} = 1$ bit

Fractional information

A dull guessing game: (submarine, MacKay p71)

Q. Is the number 36?

A. $a_1 =$ No.

$h(a_1) = \log \frac{1}{P(x\neq 36)} = \log \frac{64}{63} = 0.0227$ bits

Remember: $\log_2 x = \frac{\ln x}{\ln 2}$

Q. Is the number 42?

A. $a_2 =$ No.

$h(a_2) = \log \frac{1}{P(x\neq 42\,|\,x\neq 36)} = \log \frac{63}{62} = 0.0231$ bits

Q. Is the number 47?

A. $a_3 =$ Yes.

$h(a_3) = \log \frac{1}{P(x=47\,|\,x\neq 42, x\neq 36)} = \log \frac{62}{1} = 5.9542$ bits

Total information: $5.9542 + 0.0231 + 0.0227 = 6$ bits

Entropy

Improbable events are very informative, but don't happen very often! How much information can we *expect*?

Discrete sources:

Ensemble: $X = (x, \mathcal{A}_X, \mathcal{P}_X)$

Outcome: $x \in \mathcal{A}_x, \quad p(x\!=\!a_i) = p_i$

Alphabet: $\mathcal{A}_X = \{a_1, a_2, \ldots, a_i, \ldots a_I\}$

Probabilities: $\mathcal{P}_X = \{p_1, p_2, \ldots, p_i, \ldots p_I\}, \qquad p_i\!>\!0, \quad \sum_i p_i = 1$

Information content:

$h(x\!=\!a_i) = \log \frac{1}{p_i}, \qquad h(x) = \log \frac{1}{P(x)}$

Entropy:

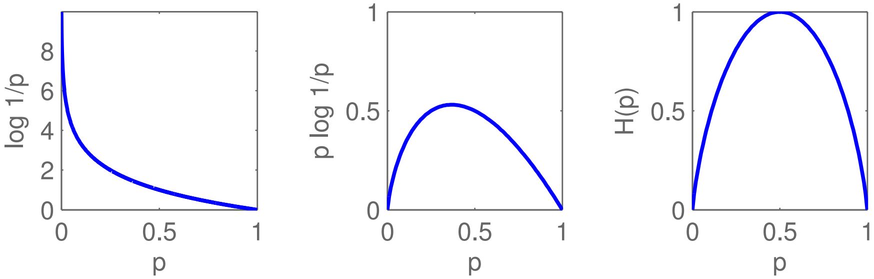
$H(X) = \sum_i p_i \log \frac{1}{p_i} = \mathbb{E}_{\mathcal{P}_X}[h(x)]$

average information content of source, also "the uncertainty of X"

Binary Entropy

Entropy of Bernoulli variable:

$$\begin{aligned} H_2(X) &= p_1 \log \frac{1}{p_1} + p_2 \log \frac{1}{p_2} \\ &= -p \log p - (1\!-\!p) \log (1\!-\!p) \end{aligned}$$



Plots take logs base 2. We define $0 \log 0 = 0$

Entropy: decomposability

Flip a coin:

Heads → A

Tails → **flip again:**

Heads → B

Tails → C

$\mathcal{A}_X = \{A,\ B,\ C\}$

$\mathcal{P}_X = \{0.5,\ 0.25,\ 0.25\}$

$H(X) = 0.5 \log \frac{1}{0.5} + 0.25 \log \frac{1}{0.25} + 0.25 \log \frac{1}{0.25} = 1.5$ bits

Or: $H(X) = H_2(0.5) + 0.5\, H_2(0.5) = 1.5$ bits

Shannon's 1948 paper §6. MacKay §2.5, p33

Why look at the decomposability of Entropy?

Mundane, but useful: it can make your algebra a lot neater.

Philosophical: we expect that the expected amount of information from a source should be the same if the same basic facts are represented in different ways and/or reported in a different order.

Shannon's paper used the desired decomposability of entropy to derive what form it must take. This is similar to how we intuited the information content from simple assumptions.

Maybe you will believe the following argument: any discrete variable could be represented as a set of binary choices. Each choice, s, cannot be compressed into less than $H_2(p_s)$ bits on average. Adding these up weighted by how often they are made gives the entropy of the original variable. So the entropy gives the limit to compressibility in general. If not convincing, we will review the full proof later (MacKay §4.2–4.6).

Where now?

Bernoulli vars. compress to $H_2(X)$ bits/symbol and no less

The entropy $H(X)$ is the compression limit on average for arbitrary random symbols. (We will gather more evidence for this later)

Where do we get the probabilities from?

How do we *actually* compress the files?
We can't explicitly list 2^{NH} items!
Can we avoid using enormous blocks?