Informatics 2D – Reasoning and Agents

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Lecture 30 – Markov Decision Processes
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Where are we?

Last time . . .

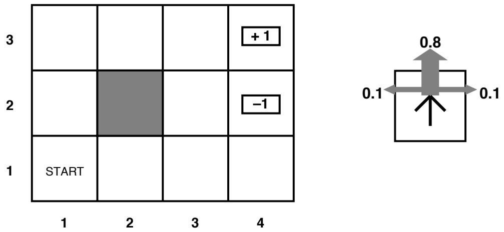
- Talked about decision making under uncertainty
- Looked at utility theory
- Discussed axioms of utility theory
- Described different utility functions
- Introduced decision networks

Today . . .

- **Markov Decision Processes**

Sequential decision problems

- So far we have only looked at one-shot decisions, but decision process are often sequential
- Example scenario: a 4x3-grid in which agent moves around (fully observable) and obtains utility of +1 or -1 in terminal states



(a) (b)

- Actions are somewhat unreliable (in deterministic world, solution would be trivial)

Markov decision processes

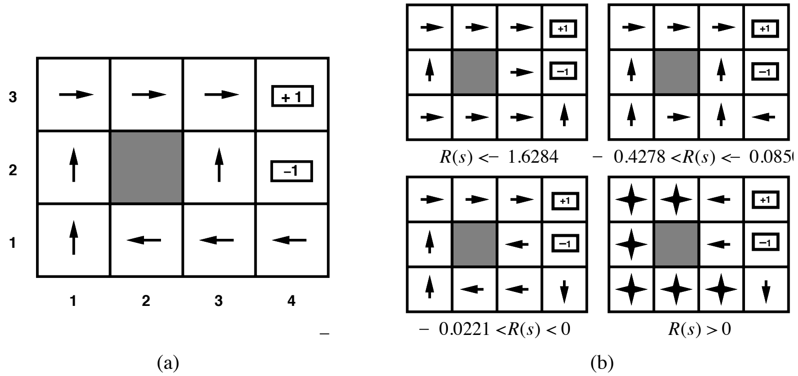
- To describe such worlds, we can use a **(transition) model** $T(s, a, s')$ denoting the probability that action a in s will lead to state s'
- Model is Markovian: probability of reaching s' depends only on s and not on history of earlier states
- Think of T as big three-dimensional table (actually a DBN)
- Utility function now depends on **environment history**
  - agent receives a reward $R(s)$ in each state s (e.g. -0.04 apart from terminal states in our example)
  - (for now) utility of environment history is the sum of state rewards
- In a sense, stochastic generalisation of search algorithms!

Markov decision processes

- Definition of a **Markov Decision Process (MDP)**:
  Initial state: S_0
  Transition model: $T(s, a, s')$
  Utility function: $R(s)$
- Solution should describe what agent does in every state
- This is called **policy**, written as π
- $\pi(s)$ for an individual state describes which action should be taken in s
- **Optimal policy** is one that yields the highest expected utility (denoted by π^*)

Example

- Optimal policies in the 4x3-grid environment
  (a) With cost of -0.04 per intermediate state π^* is conservative for (3,1)
  (b) Different cost induces direct run to terminal state/shortcut at (3,1)/no risk/avoid both exits



Optimality in sequential decision problems

- MDPs very popular in various disciplines, different algorithms for finding optimal policies
- Before we present some of them, let us look at utility functions more closely
- We have used sum of rewards as utility of environment history until now, but what are the alternatives?
- First question: **finite horizon** or **infinite horizon**
- Finite means there is a fixed time N after which nothing matters:

$$\forall k\ U_h([s_0, s_1, \ldots, s_{N+k}]) = U_h([s_0, s_1, \ldots, s_N])$$

Optimality in sequential decision problems

- This leads to **non-stationary** optimal policies (N matters)
- With infinite horizon, we get **stationary** optimal policies (time at state doesn't matter)
- We are mainly going to use infinite horizon utility functions
- NOTE: sequences to terminal states can be finite even under infinite horizon utility calculation
- Second issue: how to calculate utility of sequences
- **Stationarity** here is reasonable assumption:

$$s_0 = s_0' \wedge [s_0, s_1, s_2 \ldots] \succ [s_0', s_1', s_2', \ldots] \Rightarrow [s_1, s_2 \ldots] \succ [s_1', s_2', \ldots]$$

Optimality in sequential decision problems

- Stationarity may look harmless, but there are only two ways to assign utilities to sequences under stationarity assumptions
- **Additive rewards**:

$$U_h([s_0, s_1, s_2 \ldots]) = R(s_0) + R(S_1) + R(S_2) + \ldots$$

- **Discounted rewards** (for **discount factor** $0 \leq \gamma \leq 1$)

$$U_h([s_0, s_1, s_2 \ldots]) = R(s_0) + \gamma R(S_1) + \gamma^2 R(S_2) + \ldots$$

- Discount factor makes more distant future rewards less significant
- We will mostly use discounted rewards in what follows

Optimality in sequential decision problems

- Choosing infinite horizon rewards creates a problem
- Some sequences will be infinite with infinite (additive) reward, how do we compare them?
- Solution 1: with discounted rewards the utility is bounded if single-state rewards are

$$U_h([s_0, s_1, s_2 \ldots]) = \sum_{t=0}^{\infty} \gamma^t R(s_t) \leq \sum_{t=0}^{\infty} \gamma^t R_{max} = R_{max}/(1-\gamma)$$

- Solution 2: under **proper policies**, i.e. if agent will eventually visit terminal state, additive rewards are finite
- Solution 3: compare **average reward** per time step

Value iteration

- **Value iteration** is an algorithm for calculating optimal policy in MDPs

  Calculate the utility of each state and then select optimal action based on these utilities

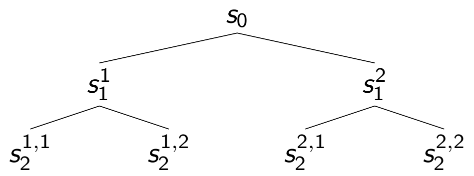
- Since discounted rewards seemed to create no problems, we will use

$$\pi^* = \arg\max_{\pi} E\left[\sum_{t=0}^{\infty} \gamma^t R(s_t) | \pi\right]$$

  as a criterion for optimal policy

Explaining $\pi^* = \arg\max_{\pi} E\left[\sum_{t=0}^{\infty} \gamma^t R(s_t) | \pi\right]$

- Each policy π yields a tree, with root node s_0, and daughters to a node s are the possible successor states given the action $\pi(s)$.
  - $T(s, a, s')$ gives the probability of traversing an arc from s to daughter s'.



- E is computed by:
  - (a) For each path p in the tree, getting the product of the (joint) probability of the path in this tree with its discounted reward, and then
  - (b) Summing over all the products from (a)
- So this is just a generalisation of single shot decision theory.

Utilities of states: : $U(s) \neq R(s)$!

- $R(s)$ is reward for being in s *now*.
- By making $U(s)$ the utility of the states that might follow it, $U(s)$ captures *long-term* advantages from being in s

  $U(s)$ reflects what you can do *from s;*
  $R(s)$ does not.

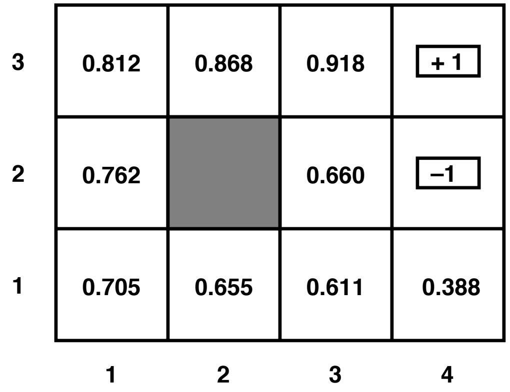
- States that follow depend on π. So utility of s given π is:

$$U^{\pi}(s) = E\left[\sum_{t=0}^{\infty} \gamma^t R(s_t) | \pi, s_0 = s\right]$$

- With this, "true" utility $U(s)$ is $U^{\pi^*}(s)$ (expected sum of discounted rewards if executing optimal policy)

Utilities in our example

- $U(s)$ computed for our example from algorithms to come.
- $\gamma = 1$, $R(s) = -0.04$ for nonterminals.



	1	2	3	4
3	0.812	0.868	0.918	+1
2	0.762		0.660	–1
1	0.705	0.655	0.611	0.388

Utilities of states

- Given $U(s)$, we can easily determine optimal policy:

$$\pi^*(s) = \arg\max_a \sum_{s'} T(s, a, s')U(s')$$

- Direct relationship between utility of a state and that of its neighbours:

  Utility of a state is immediate reward plus expected utility of subsequent states if agent chooses optimal action

- This can be written as the famous **Bellman equations**:

$$U(s) = R(s) + \gamma \max_a \sum_{s'} T(s, a, s')U(s')$$

The value iteration algorithm

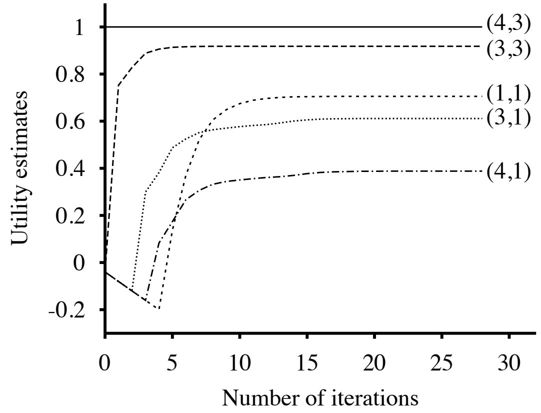
- For n states we have n Bellman equations with n unknowns (utilities of states)
- Value iteration is an iterative approach to solving the n equations.
- Start with arbitrary values and update them as follows:

$$U_{i+1}(s) \leftarrow R(s) + \gamma \max_a \sum_{s'} T(s, a, s')U_i(s')$$

- The algorithm converges to right and unique solution
- Like propagating values through network or utilities

The value iteration algorithm

- Value iteration in our example: evolution of utility values of states

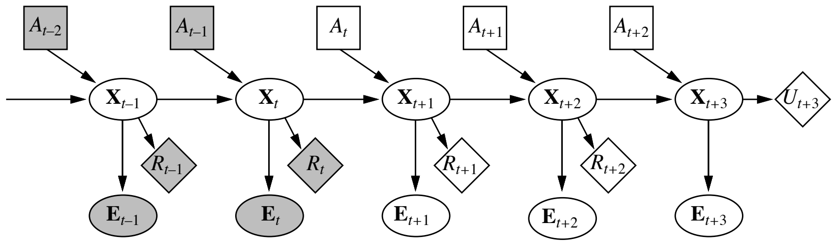


Decision-theoretic agents

- We now have (tediously) gathered all the ingredients to build decision-theoretic agents
- Transition and observation models will be described by a DBN
- They will be augmented by decision and utility nodes to obtain a **dynamic DN**
- Decisions will be made by projecting forward possible action sequences and choosing the best one
- Practical design for a **utility-based agent**

Decision-theoretic agents

- Dynamic decision networks look something like this
- General form of everything we have talked about in uncertainty part



Summary

- Sequential decision making
- Defined MDPs to model stochastic multi-step decision making processes
- Value iteration and policy iteration algorithms
- Design of decision-theoretic utility-based agents based on DDNs
- Completes our account of reasoning under uncertainty