Informatics 2D – Reasoning and Agents

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Alex Lascarides
alex@inf.ed.ac.uk

Lecture 27 – Time and Uncertainty II

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Where are we?

Last time ...

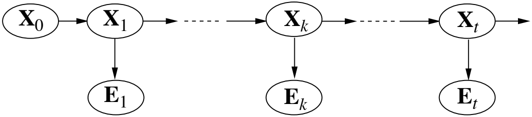
- Time in reasoning about uncertainty
- Markov assumption, stationarity
- Algorithms for reasoning about temporal processes
- Filtering and prediction

Today ...

- **Time and uncertainty II**

Smoothing

- Smoothing is computation of distribution of past states given current evidence, i.e. $\mathbf{P}(\mathbf{X}_k|\mathbf{e}_{1:t})$, $1 \leq k < t$



- Easiest to view as 2-step process (up to k, then $k+1$ to t)

$$\begin{aligned}\mathbf{P}(\mathbf{X}_k|\mathbf{e}_{1:t}) &= \mathbf{P}(\mathbf{X}_k|\mathbf{e}_{1:k}, \mathbf{e}_{k+1:t}) \\ &= \alpha \mathbf{P}(\mathbf{X}_k|\mathbf{e}_{1:k})\mathbf{P}(\mathbf{e}_{k+1:t}|\mathbf{X}_k, \mathbf{e}_{1:k}) && \text{(Bayes' rule)} \\ &= \alpha \mathbf{P}(\mathbf{X}_k|\mathbf{e}_{1:k})\mathbf{P}(\mathbf{e}_{k+1:t}|\mathbf{X}_k) && \text{(conditional independence)} \\ &= \alpha \mathbf{f}_{1:k}\mathbf{b}_{k+1:t}\end{aligned}$$

- Here "backward" message is $\mathbf{b}_{k+1:t} = \mathbf{P}(\mathbf{e}_{k+1:t}|\mathbf{X}_k)$ analogous to forward message

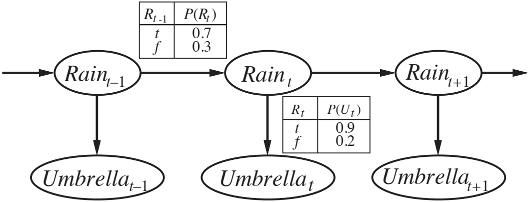
Smoothing

- Formula for backward message:

$$\mathbf{P}(\mathbf{e}_{k+1:t}|\mathbf{X}_k) = \sum_{\mathbf{x}_{k+1}} P(\mathbf{e}_{k+1}|\mathbf{x}_{k+1})P(\mathbf{e}_{k+2:t}|\mathbf{x}_{k+1})\mathbf{P}(\mathbf{x}_{k+1}|\mathbf{X}_k)$$

- First term is sensor model; third term is transition model; second is 'recursive call'
- Define $\mathbf{b}_{k+1:t} = \text{BACKWARD}(\mathbf{b}_{k+2:t}, \mathbf{e}_{k+1:t})$
- The backward phase has to be initialised with $\mathbf{b}_{t+1:t} = \mathbf{P}(\mathbf{e}_{t+1:t}|\mathbf{X}_t) = \mathbf{1}$ (a vector of 1s) because probability of observing empty sequence is 1
- As before, all this is quite abstract, back to our example

Umbrella World: Compute $\mathbf{P}(R_1|u_1, u_2)$



We have $\mathbf{P}(R_1|u_1, u_2) = \alpha \mathbf{P}(R_1|u_1)\mathbf{P}(u_2|R_1)$
So we'll need to remind ourselves of $\mathbf{P}(R_1|u_1)$ from last lecture:

- $\mathbf{P}(R_1) = \sum_{r_0} \mathbf{P}(R_1|r_0)P(r_0) = \langle 0.7, 0.3\rangle \times 0.5 + \langle 0.3, 0.7\rangle \times 0.5 = \langle 0.5, 0.5\rangle$
- Update with evidence $U_1 = true$ yields:

$$\mathbf{P}(R_1|u_1) = \alpha \mathbf{P}(u_1|R_1)\mathbf{P}(R_1) = \alpha\langle 0.9, 0.2\rangle\langle 0.5, 0.5\rangle \approx \langle 0.818, 0.182\rangle$$

Smoothing Example Continued

$\mathbf{P}(R_1|u_1, u_2) = \alpha\mathbf{P}(R_1|u_1)\mathbf{P}(u_2|R_1)$

- Forward filtering process yielded $\langle 0.818, 0.182\rangle$ for first term
- The second term can be obtained through backward recursion:

$$\mathbf{P}(u_2|R_1) = \sum_{r_2} P(u_2|r_2)P(|r_2)\mathbf{P}(r_2|R_1)$$
$$= (0.9 \times 1 \times \langle 0.7, 0.3\rangle) + (0.2 \times 1 \times \langle 0.3, 0.7\rangle) = \langle 0.69, 0.41\rangle$$

- Plugged into the above equation this yields

$$\mathbf{P}(R_1|u_1, u_2) = \alpha\langle 0.818, 0.182\rangle \times \langle 0.69, 0.41\rangle \approx \langle 0.883, 0.117\rangle$$

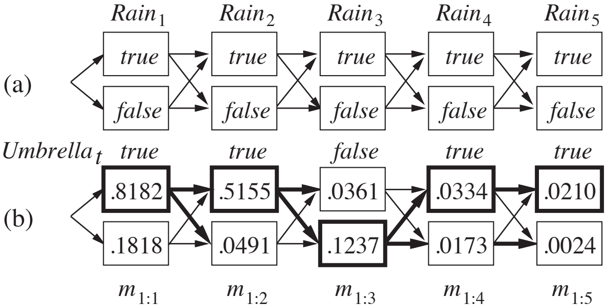
- So our confidence that it rained on Day 1 increases when we see the umbrella on the second day as well as the first.
- A simple improved version of this that stores results runs in linear time (**forward-backward** algorithm)

Finding the most likely sequence

- Suppose [*true*, *true*, *false*, *true*, *true*] is the umbrella sequence for first five days, what is the most likely weather sequence that caused it?
- Could use smoothing procedure to find posterior distribution for weather at each step and then use most likely weather at each step to construct sequence
- NO! Smoothing considers distributions over individual time steps, but we must consider **joint** probabilities over all time steps
- Actual algorithm is based on viewing each sequence as path through a graph (nodes=states at each time step)

Finding the most likely sequence

- In umbrella example:



- Look at states with $Rain_5 = true$ (part (a)), Markov property
  - most likely path to this state consists of most likely path to state at time 4 followed by transition to $Rain_5 = true$
  - state at time 4 that will become part of the path is whichever maximises likelihood of the path

Finding the most likely sequence

- There is a recursive relationship between most likely paths to $\mathbf{x}_{t+1}$ and most likely paths to each state $\mathbf{x}_t$

$$\max_{\mathbf{x}_1 \ldots \mathbf{x}_t} \mathbf{P}(\mathbf{x}_1, \ldots, \mathbf{x}_t, \mathbf{X}_{t+1} | \mathbf{e}_{1:t+1})$$
$$= \alpha \mathbf{P}(\mathbf{e}_{t+1} | \mathbf{X}_{t+1}) \max_{\mathbf{x}_t} (\mathbf{P}(\mathbf{X}_{t+1} | \mathbf{x}_t) \max_{\mathbf{x}_1 \ldots \mathbf{x}_{t-1}} P(\mathbf{x}_1, \ldots, \mathbf{x}_{t-1}, \mathbf{x}_t | \mathbf{e}_{1:t}))$$

- This is like filtering only that the forward message is replaced by

$$\mathbf{m}_{1:t} = \max_{\mathbf{x}_1 \ldots \mathbf{x}_{t-1}} \mathbf{P}(\mathbf{x}_1, \ldots, \mathbf{x}_{t-1}, \mathbf{X}_t | \mathbf{e}_{1:t})$$

- And summation is now replaced by maximisation

Finding the most likely sequence

- This algorithm (**Viterbi algorithm**) is similar to filtering
- Runs forward along sequence computing **m** message in each step
- Progress in example shown in part (b) of diagram above
- In the end it has probability for most likely sequence for reaching each final state
  Easy to determine overall most likely sequence
- Has to keep pointers from each state back to the best state that leads to it

Hidden Markov Models

- So far, we have seen a general model for temporal probabilistic reasoning (independent of transition/sensor models)
- In this and the following lecture we are going to look at more concrete models and applications
- **Hidden Markov Models (HMMs)**: temporal probabilistic model in which state of the process is described by a single variable
- Like our umbrella example (single variable $Rain_t$)
- More than one variable can be accommodated, but only by combining them into a single "mega-variable"
- Structure of HMMs allows for a very simple and elegant matrix implementation of basic algorithms

Summary

- The forward-backward algorithm
- Finding the most likely sequence (Viterbi algorithm)
- Talked about HMMs
- HMMs: single state variable, simplifies algorithms (see other courses for these)
- Huge significance, for example in speech recognition:

$$P(\mathit{words}|\mathit{signal}) = \alpha P(\mathit{signal}|\mathit{words}) P(\mathit{words})$$

- Vast array of applications, but also limits.
- Next time: **Dynamic Bayesian Networks**