Informatics 2D – Reasoning and Agents

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Lecture 24 – Exact Inference in Bayesian Networks
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Where are we?

Last time ...

- Introduced Bayesian networks
- Allow for compact representation of JPDs
- Methods for efficient representations of CPTs
- But how hard is inference in BNs?

Today ...

- **Inference in Bayesian networks**

Inference in BNs

- Basic task: compute posterior distribution for set of **query variables** given some observed **event** (i.e. assignment of values to **evidence variables**)
- Formally: determine $\mathbf{P}(X|\mathbf{e})$ given query variables $\mathbf{X}$, evidence variables $\mathbf{E}$ (and non-evidence or **hidden** variables $\mathbf{Y}$)
- Example:
  $\mathbf{P}(Burglary|JohnCalls = true, MaryCalls = true) = \langle 0.284, 0.716\rangle$
- First we will discuss exact algorithms for computing posterior probabilities then approximate methods later

Inference by enumeration

- We have seen that any conditional probability can be computed from a full JPD by summing terms
- $\mathbf{P}(X|\mathbf{e}) = \alpha\mathbf{P}(X, \mathbf{e}) = \alpha \sum_{\mathbf{y}} \mathbf{P}(X, \mathbf{e}, \mathbf{y})$
- Since BN gives complete representation of full JPD, we must be able to answer a query by computing sums of products of conditional probabilities from the BN
- Consider query $\mathbf{P}(Burglary|JohnCalls = true, MaryCalls = true) = \mathbf{P}(B|j, m)$
- $\mathbf{P}(B|j, m) = \alpha\mathbf{P}(B, j, m) = \alpha \sum_e \sum_a \mathbf{P}(B, e, a, j, m)$

Inference by enumeration

- Recall $P(x_1, \ldots, x_n) = \prod_{i=1}^{n} P(x_i | parents(X_i))$
- We can use CPTs to simplify this exploiting BN structure
- For $Burglary = true$:

$$P(b|j,m) = \alpha \sum_{e} \sum_{a} P(b)P(e)P(a|b,e)P(j|a)P(m|a)$$

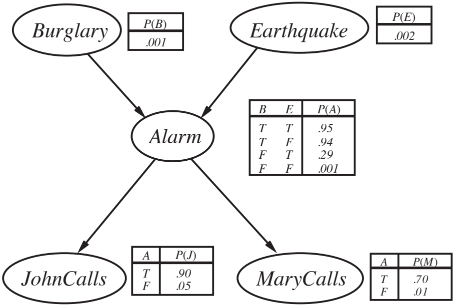
- But we can improve efficiency of this by moving terms outside that don't depend on sums

$$P(b|j,m) = \alpha P(b) \sum_{e} P(e) \sum_{a} P(a|b,e)P(j|a)P(m|a)$$

- To compute this, we need to loop through variables in order and multiply CPT entries; for each summation we need to loop over variable's possible values

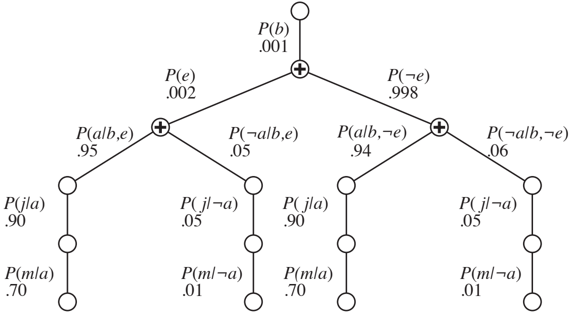
Example

- New burglar alarm has been fitted, fairly reliable but sometimes reacts to earthquakes
- Neighbours John and Mary promise to call when they hear alarm
- John sometimes mistakes phone for alarm, and Mary listens to loud music and sometimes doesn't hear alarm



The variable elimination algorithm

- Enumeration method is computationally quite hard.
- You often compute the same thing several times;
  e.g. $P(j|a)P(m|a)$ and $P(j|\neg a)P(m|\neg a)$ for each value of e
- Evaluation of expression shown in the following tree:



The variable elimination algorithm

- Idea of **variable elimination**: avoid repeated calculations
- Basic idea: store results after doing calculation once
- Works bottom-up by evaluating subexpressions
- Assume we want to evaluate

$$\mathbf{P}(B|j,m) = \alpha \underbrace{\mathbf{P}(B)}_{\mathbf{f}_1(B)} \sum_e \underbrace{\mathbf{P}(e)}_{\mathbf{f}_2(E)} \sum_a \underbrace{\mathbf{P}(a|B,e)}_{\mathbf{f}_3(A,B,E)} \underbrace{P(j|a)}_{\mathbf{f}_4(A)} \underbrace{P(m|a)}_{\mathbf{f}_5(A)}$$

- We've annotated each part with a **factor**.
- A factor is a **matrix**, indexed with its argument variables. E.g:
  - Factor $\mathbf{f}_5(A)$ corresponds to $P(m|a)$ and depends just on A because m is fixed (it's a 2×1 matrix).

$$\mathbf{f}_5(A) = \langle P(m|a), P(m|\neg a) \rangle$$

  - $\mathbf{f}_3(A, B, E)$ is a $2 \times 2 \times 2$ matrix for $\mathbf{P}(a|B, e)$

The variable elimination algorithm

$\mathbf{P}(B|j,m) = \alpha \mathbf{f}_1(B) \times \sum_e \mathbf{f}_2(E) \sum_a \mathbf{f}_3(A,B,E) \times \mathbf{f}_4(A) \times \mathbf{f}_5(A)$

- Summing out A produces a 2×2 matrix (via **pointwise product**):
$$\begin{aligned}\mathbf{f}_6(B,E) &= \sum_a \mathbf{f}_3(A,B,E) \times \mathbf{f}_4(A) \times \mathbf{f}_5(A) \\ &= (\mathbf{f}_3(a,B,E) \times \mathbf{f}_4(a) \times \mathbf{f}_5(a)) + \\ &\quad\ (\mathbf{f}_3(\neg a,B,E) \times \mathbf{f}_4(\neg a) \times \mathbf{f}_5(\neg a))\end{aligned}$$
- So now we have
$\mathbf{P}(B|j,m) = \alpha \mathbf{f}_1(B) \times \sum_e \mathbf{f}_2(E) \times \mathbf{f}_6(B,E)$
- Sum out E in the same way:
$\mathbf{f}_7(B) = (\mathbf{f}_2(e) \times \mathbf{f}_6(B,e)) + (\mathbf{f}_2(\neg e) \times \mathbf{f}_6(B,\neg e))$
- Using $\mathbf{f}_1(B) = \mathbf{P}(B)$, we can finally compute
$$\mathbf{P}(B|j,m) = \alpha \mathbf{f}_1(B) \times \mathbf{f}_7(B)$$
- Remains to define pointwise product and summing out

An example

- Pointwise product yields product for union of variables in its arguments:

$$\mathbf{f}(X_1 \ldots X_i, Y_1 \ldots Y_j, Z_1 \ldots Z_k) = \mathbf{f}_1(X_1 \ldots X_i, Y_1 \ldots Y_j)\mathbf{f}_2(Y_1 \ldots Y_j, Z_1 \ldots Z_k)$$

A	B	$\mathbf{f}_1(A,B)$	B	C	$\mathbf{f}_2(B,C)$	A	B	C	$\mathbf{f}(A,B,C)$
T	T	0.3	T	T	0.2	T	T	T	0.3×0.2
T	F	0.7	T	F	0.8	T	T	F	0.3×0.8
F	T	0.9	F	T	0.6	T	F	T	0.7×0.6
F	F	0.1	F	F	0.4	T	F	F	0.7×0.4
						F	T	T	0.9×0.2
						F	T	F	0.9×0.8
						F	F	T	0.1×0.6
						F	F	F	0.1×0.4

- For example $\mathbf{f}(T,T,F) = \mathbf{f}_1(T,T) \times \mathbf{f}_2(T,F)$

An example

- Summing out is similarly straightforward
- Trick: any factor that does not depend on the variable to be summed out can be moved outside the summation process
- For example

$$\sum_e \mathbf{f}_2(E) \times \mathbf{f}_3(A,B,E) \times \mathbf{f}_4(A) \times \mathbf{f}_5(A)$$

$$= \mathbf{f}_4(A) \times \mathbf{f}_5(A) \times \sum_e \mathbf{f}_2(E) \times \mathbf{f}_3(A,B,E)$$

- Matrices are only multiplied when we need to sum out a variable from the accumulated product

Another Example: $\mathbf{P}(J|b) = \langle P(j|b), P(\neg j|b)\rangle$

$$
\begin{aligned}
\mathbf{P}(J|b) &= \alpha \textstyle\sum_e \sum_a \sum_m \mathbf{P}(J,b,e,a,m) && \text{prod., marg.}\\
&= \alpha \textstyle\sum_e \sum_a \sum_m P(b)P(e)P(a|b,e)\mathbf{P}(J|a)P(m|a) && \text{cond. indep.}\\
&= \alpha' \textstyle\sum_e \underbrace{P(e)}_{\mathbf{f}_1(E)} \sum_a \underbrace{P(a|b,e)}_{\mathbf{f}_2(A,E)} \underbrace{\mathbf{P}(J|a)}_{\mathbf{f}_3(J,A)} \underbrace{\sum_m P(m|a)}_{=1} && \text{move terms}\\
&= \alpha' \textstyle\sum_e \underset{2\times 1}{\mathbf{f}_1(E)} \sum_a \underset{2\times 2}{\mathbf{f}_2(A,E)} \underset{2\times 2}{\mathbf{f}_3(J,A)}\\
&= \alpha' \textstyle\sum_e \underset{2\times 1}{\mathbf{f}_1(E)} \underset{2\times 2}{\mathbf{f}_4(J,E)}\\
&= \alpha' \mathbf{f}_5(J)
\end{aligned}
$$

Can eliminate all variables that aren't ancestors of query or evidence variables!

Summary

- Inference in Bayesian Networks
- Exact methods: enumeration, variable elimination algorithm
- Computationally intractable in the worst case
- Next time: **Approximate inference in Bayesian Networks**