Inf2b - Learning

Lecture 8: Real-valued distributions and Gaussians

Hiroshi Shimodaira
(Credit: Iain Murray and Steve Renals)

Centre for Speech Technology Research (CSTR)
School of Informatics
University of Edinburgh

http://www.inf.ed.ac.uk/teaching/courses/inf2b/
https://piazza.com/ed.ac.uk/spring2020/infr08028
Office hours: Wednesdays at 14:00-15:00 in IF-3.04

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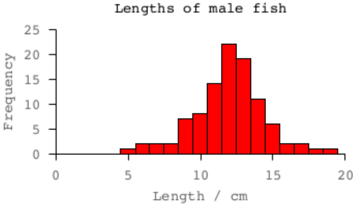
Today's Schedule

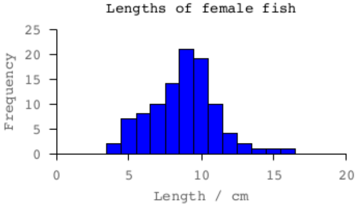
Real-valued distributions and Gaussians

1. Continuous random variables
2. The Gaussian distribution (one-dimensional)
3. Maximum likelihood estimation
4. The multidimensional Gaussian distribution

Discrete to continuous random variables

Fish example again:



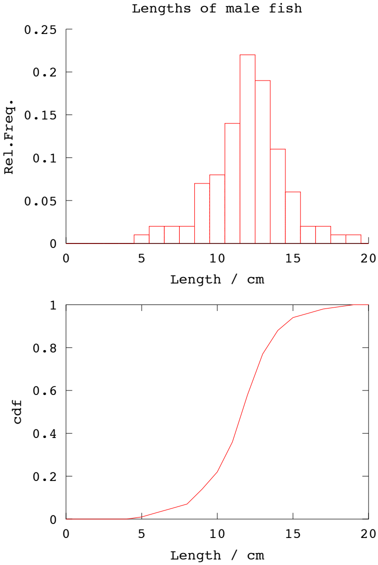


$$c^* = \arg\max_c P(c|x) = \arg\max_c \frac{P(x|c)P(c)}{P(x)} = \arg\max_c P(x|c)P(c)$$

- What if the number of bins $\to \infty$? (i.e. the width of bin $\to 0$)
- $P(X = x|C)$ will be almost 0 everywhere!
- We instead consider a cumulative distribution function (cdf) with a continuous random variable:

$$F(x) = P(X \leq x)$$

Cumulative distribution functions graphed



Cumulative distribution function properties

Cumulative distribution functions have the following properties:

1. $F(-\infty) = 0$;
2. $F(\infty) = 1$;
3. If $a \leq b$ then $F(a) \leq F(b)$.

To obtain the probability of falling in an interval we can do the following:

$$
\begin{aligned}
P(a < X \leq b) &= P(X \leq b) - P(X \leq a) \\
&= F(b) - F(a)
\end{aligned}
$$

Probability density function (pdf)

- The rate of change of the cdf gives us the probability density function (pdf) , $p(x)$:

$$p(x) = \frac{d}{dx} F(x) = F'(x)$$

$$F(x) = \int_{-\infty}^{x} p(x)\, dx$$

- $p(x)$ is **not** the probability that X has value x. But the pdf is proportional to the probability that X lies in a small interval $[x, x + dx]$.
- Notation: p for pdf, P for probability

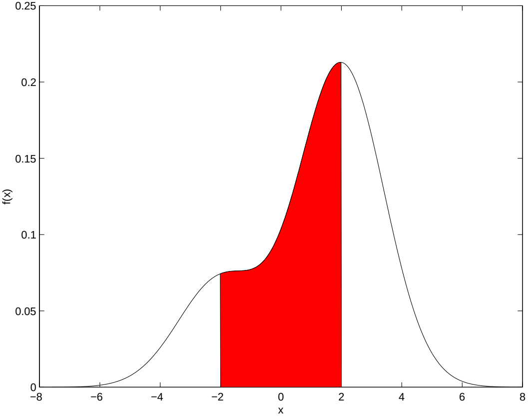
pdf and cdf

The probability that the random variable lies in interval (a, b) is given by:

$$
\begin{aligned}
P(a < X \leq b) &= F(b) - F(a) \\
&= \int_{-\infty}^{b} p(x)\, dx - \int_{-\infty}^{a} p(x)\, dx \\
&= \int_{a}^{b} p(x)\, dx
\end{aligned}
$$

pdf and cdf

The probability that the random variable lies in interval (a, b) is the area under the pdf between a and b:



The Gaussian distribution

- The Gaussian (or Normal) distribution is the most common (and easily analysed) continuous distribution
- It is also a reasonable model in many situations (the famous "bell curve")
- If a (scalar) variable has a Gaussian distribution, then it has a probability density function with this form:

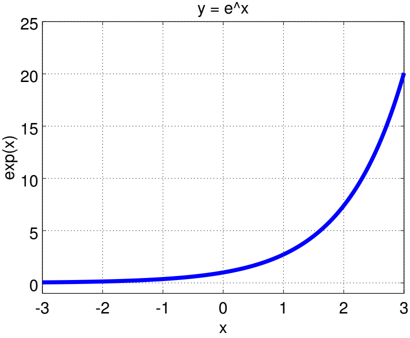
$$p(x \mid \mu, \sigma^2) = N(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$$

NB: $\exp(f(x)) = e^{f(x)}$

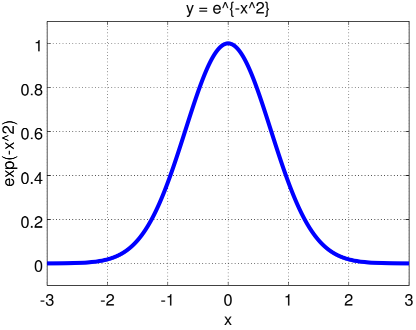
- The Gaussian is described by two parameters:
  - the mean μ (location)
  - the variance σ^2 (dispersion)

Natural exponential function

$$y = e^x = \exp(x)$$

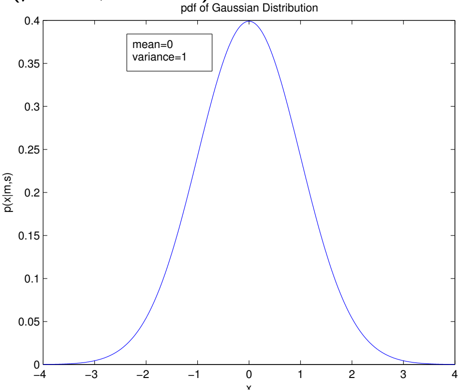


$$y = \exp(-x^2)$$



Plot of Gaussian distribution

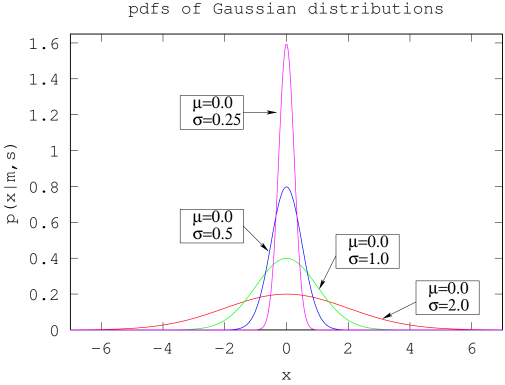
- Gaussians have the same shape, with the location controlled by the mean, and the spread controlled by the variance
- One-dimensional Gaussian with zero mean and unit variance ($\mu = 0, \sigma^2 = 1$)



Another plot of a Gaussian

Properties of the Gaussian distribution

$$N(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2} \right)$$



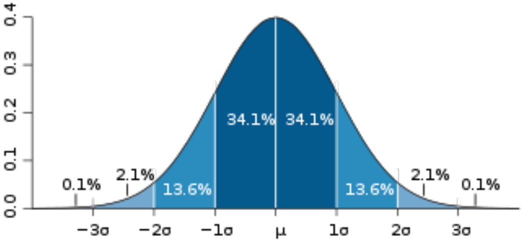
$$\int_{-\infty}^{\infty} N(x; \mu, \sigma^2) dx = 1$$

$$\lim_{\sigma \to 0} N(x; \mu, \sigma^2) = \delta(x - \mu)$$

(Dirac delta function)

Facts about the Gaussian distribution

- A Gaussian can be used to describe approximately any random variable that tends to cluster around the mean
- Concentration:
  - About 68% of values drawn from a normal distribution are within one SD away from the mean
  - About 95% are within two SDs
  - About 99.7% lie within three SDs of the mean



Central Limit Theorem

- Under certain conditions, the sum of a large number of random variables will have approximately normal distribution.
- Several other distributions are well approximated by the Normal distribution:
  - Binomial $B(n, p)$, when n is large and p is not too close to 1 or 0
  - Poisson $P_o(\lambda)$ when λ is large
  - Other distributions including chi-squared and Student's T
- The Wikipedia entry on the Gaussian distribution is good

Parameter estimation form data

- Estimate the mean and variance parameters of a Gaussian from data $\{x_1, x_2, \ldots, x_N\}$
- Sample mean and sample variance (unbiased) estimates:

$$\hat{\mu} = \frac{1}{N} \sum_{n=1}^{N} x_n$$

$$\hat{\sigma}^2 = \frac{1}{N-1} \sum_{n=1}^{N} (x_n - \hat{\mu})^2$$

- Maximum likelihood estimates (MLE):

$$\hat{\mu}_{\text{ML}} = \frac{1}{N} \sum_{n=1}^{N} x_n$$

$$\hat{\sigma}^2_{\text{ML}} = \frac{1}{N} \sum_{n=1}^{N} (x_n - \hat{\mu}_{\text{ML}})^2$$

Example: Gaussians

A pattern recognition problem has two classes, S and T. Some observations are available for each class:

Class S	10	8	10	10	11	11
Class T	12	9	15	10	13	13

The mean and variance of each pdf are estimated with MLE.

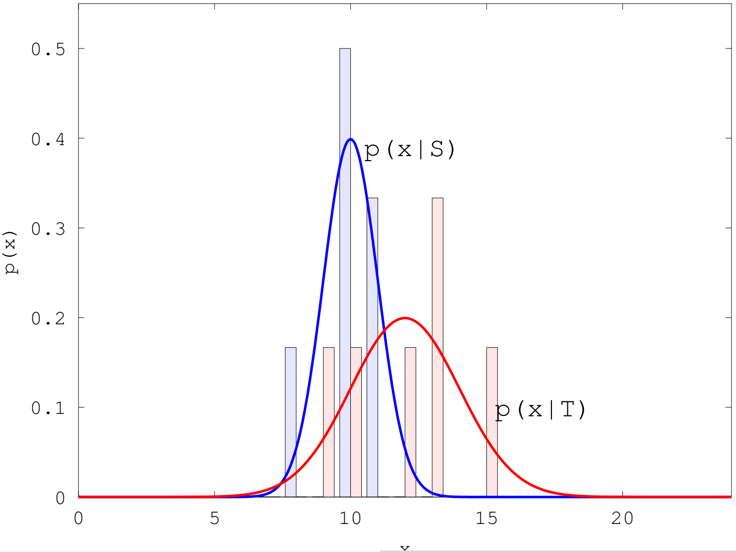
$$\begin{aligned} S: &\quad \text{mean} = 10; \quad \text{variance} = 1 \\ T: &\quad \text{mean} = 12; \quad \text{variance} = 4 \end{aligned}$$

$$p(x|S) = \frac{1}{\sqrt{2\pi \cdot 1}} \exp\left(-\frac{(x-10)^2}{2 \cdot 1}\right)$$

$$p(x|T) = \frac{1}{\sqrt{2\pi \cdot 4}} \exp\left(-\frac{(x-12)^2}{2 \cdot 4}\right)$$

Example: Gaussians *(cont.)*

Sketch the pdf for each class. cf. the histograms



Parameter estimation as an optimisation problem

- Given an observation (training) set of N samples:
  $\mathcal{D} = \{x_1, x_2, \ldots, x_N\}$
- How can we estimate the mean μ and variance σ^2 of the population?
- Define the problem as an optimisation problem

Maximum Likelihood (ML) estimation:

$$\max_{\mu,\sigma^2} p(\mathcal{D} \mid \mu, \sigma^2)$$

NB: ML is just a one criterion for parameter estimation

ML estimation of a univariate Gaussian pdf

Assumption:

Samples $\mathcal{D} = \{x_n\}_{n=1}^{N}$ are drawn independently from the same distribution (i.i.d.)

Likelihood:

$$\begin{aligned} p(\mathcal{D} \mid \mu, \sigma^2) &= p(x_1, \ldots, x_N \mid \mu, \sigma^2) \\ &= p(x_1 \mid \mu, \sigma^2) \cdots p(x_N \mid \mu, \sigma^2) = \prod_{n=1}^{N} p(x_n \mid \mu, \sigma^2) \\ &= L(\mu, \sigma^2 \mid \mathcal{D}) \end{aligned}$$

Optimisation problem:

Find such parameters μ and σ^2 that maximise the likelihood:

$$\max_{\mu, \sigma^2} L(\mu, \sigma^2 \mid \mathcal{D})$$

The log likelihood: NB: the natural log (ln) is assumed

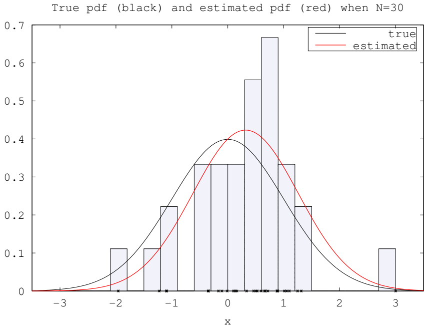
$$
\begin{aligned}
LL(\mu, \sigma^2 \mid \mathcal{D}) = \ln L(\mu, \sigma^2 \mid \mathcal{D}) &= \ln \prod_{n=1}^{N} p(x_n \mid \mu, \sigma^2) \\
&= \sum_{n=1}^{N} \ln p(x_n \mid \mu, \sigma^2) \\
&= \sum_{n=1}^{N} \ln \left(\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(x_n - \mu)^2}{2\sigma^2} \right) \right) \\
&= -\frac{N}{2} \ln(2\pi) - \frac{N}{2} \ln\left(\sigma^2\right) - \sum_{n=1}^{N} \frac{(x_n - \mu)^2}{2\sigma^2}
\end{aligned}
$$

ML estimation of a univariate Gaussian pdf *(cont.)*

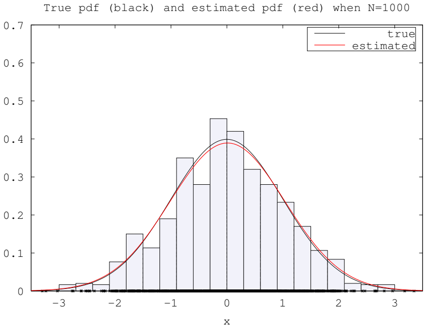
$$LL(\mu, \sigma^2 \,|\, \mathcal{D}) = -\frac{N}{2} \ln(2\pi) - \frac{N}{2} \ln\left(\sigma^2\right) - \sum_{n=1}^{N} \frac{(x_n - \mu)^2}{2\sigma^2}$$

$$\begin{aligned}
\frac{\partial LL(\mu, \sigma^2 \,|\, \mathcal{D})}{\partial \mu} &= 2 \sum_{n=1}^{N} \frac{x_n - \mu}{2\sigma^2} = 0 \\
&\Rightarrow \hat{\mu} = \frac{1}{N} \sum_{n=1}^{N} x_n \\
\frac{\partial LL(\hat{\mu}, \sigma^2 \,|\, \mathcal{D})}{\partial \sigma^2} &= -\frac{N}{2} \frac{1}{\sigma^2} + \sum_{n=1}^{N} \frac{(x_n - \hat{\mu})^2}{2(\sigma^2)^2} = 0 \\
&\Rightarrow \sigma^2 = \frac{1}{N} \sum_{n=1}^{N} (x_n - \hat{\mu})^2
\end{aligned}$$

Examples of parameter estimation with MLE



$N = 30$



$N = 1000$

The multidimensional Gaussian distribution

- The D-dimensional vector $\boldsymbol{x} = (x_1, \ldots, x_D)^T$ is multivariate Gaussian if it has a probability density function of the following form:

  $$p(\mathbf{x} \,|\, \boldsymbol{\mu}, \boldsymbol{\Sigma}) = \frac{1}{(2\pi)^{D/2} |\boldsymbol{\Sigma}|^{1/2}} \exp\left(-\frac{1}{2} (\mathbf{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right).$$

  The pdf is parameterised by the mean vector $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_D)^T$ and the covariance matrix $\boldsymbol{\Sigma} = (\sigma_{ij})$.
- The 1-dimensional Gaussian is a special case of this pdf
- The argument to the exponential $\frac{1}{2} (\boldsymbol{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{x} - \boldsymbol{\mu})$ is referred to as a *quadratic form*.

Covariance matrix

- The mean vector $\boldsymbol{\mu}$ is the expectation of $\boldsymbol{x}$:

  $$\boldsymbol{\mu} = E[\mathbf{x}]$$

- The covariance matrix $\boldsymbol{\Sigma}$ is the expectation of the deviation of $\boldsymbol{x}$ from the mean:

  $$\boldsymbol{\Sigma} = E[(\mathbf{x} - \boldsymbol{\mu})(\mathbf{x} - \boldsymbol{\mu})^T]$$

- $\boldsymbol{\Sigma}$ is a $D \times D$ symmetric matrix: $\boldsymbol{\Sigma}^T = \boldsymbol{\Sigma}$

  $$\sigma_{ij} = E[(x_i - \mu_i)(x_j - \mu_j)] = E[(x_j - \mu_j)(x_i - \mu_i)] = \sigma_{ji} \,.$$

- The sign of the covariance σ_{ij} helps to determine the relationship between two components:
  - If x_j is large when x_i is large, then $(x_j - \mu_j)(x_i - \mu_i)$ will tend to be positive;
  - If x_j is small when x_i is large, then $(x_j - \mu_j)(x_i - \mu_i)$ will tend to be negative.

Covariance matrix *(cont.)*

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \cdots & \cdots & \sigma_{1D} \\ \sigma_{21} & \sigma_{22} & \cdots & \cdots & \cdots & \sigma_{2D} \\ \vdots & \vdots & \ddots & & & \vdots \\ \vdots & \vdots & & \sigma_{ii} & & \vdots \\ \vdots & \vdots & & & \ddots & \vdots \\ \sigma_{D1} & \sigma_{D2} & \cdots & \cdots & \cdots & \sigma_{DD} \end{pmatrix}$$

- $\sigma_i^2 = \sigma_{ii}$
- $|\mathbf{\Sigma}| = \det(\mathbf{\Sigma})$: determinant
  e.g. for $D = 2$,
  $$|\mathbf{\Sigma}| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = a \times d - b \times c$$
- See dimensionality reduction with PCA in Lecture Slides (3).

Parameter estimation

Maximum likelihood estimation (MLE):

$$\boldsymbol{\mu} = E[\mathbf{x}]$$

$$\hat{\boldsymbol{\mu}}_{\text{ML}} = \frac{1}{N} \sum_{n=1}^{N} \boldsymbol{x}_n$$

$$\boldsymbol{\Sigma} = E[(\mathbf{x} - \boldsymbol{\mu})(\mathbf{x} - \boldsymbol{\mu})^T]$$

$$\hat{\boldsymbol{\Sigma}}_{\text{ML}} = \frac{1}{N} \sum_{n=1}^{N} (\boldsymbol{x}_n - \hat{\boldsymbol{\mu}}_{\text{ML}})(\boldsymbol{x}_n - \hat{\boldsymbol{\mu}}_{\text{ML}})^T$$

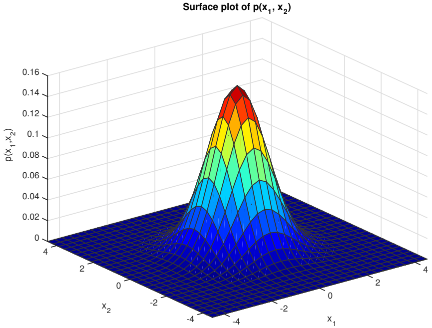
Correlation matrix

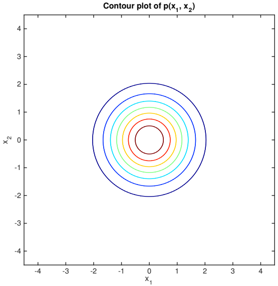
The covariance matrix is not scale-independent: Define the correlation matrix R of correlation coefficients ρ_{ij}:

$$R = (\rho_{ij})$$
$$\rho_{ij} = \frac{\sigma_{ij}}{\sqrt{\sigma_{ii}\sigma_{jj}}}$$
$$\rho_{ii} = 1$$

- Scale-independent (ie independent of the measurement units) and location-independent, ie:
$$\rho(x_i, x_j) = \rho(ax_i + b, cx_j + d) \qquad \text{for } a > 0, c > 0$$
- The correlation coefficient satisfies $-1 \leq \rho \leq 1$, and
$$\rho(x, y) = +1 \qquad \text{if } y = ax + b \quad a > 0$$
$$\rho(x, y) = -1 \qquad \text{if } y = ax + b \quad a < 0$$

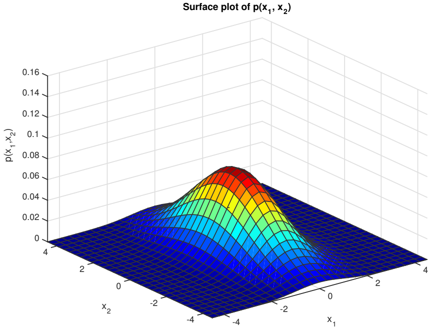
Spherical Gaussian

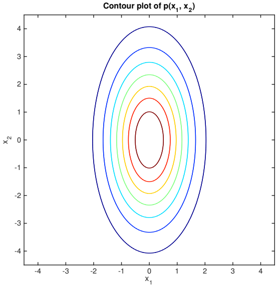




$$\boldsymbol{\mu} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \boldsymbol{\Sigma} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

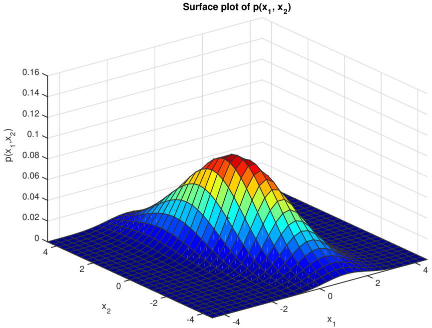
2-D Gaussian with a diagonal covariance matrix

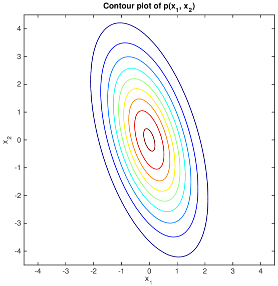




$$\boldsymbol{\mu} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \boldsymbol{\Sigma} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \qquad R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

2-D Gaussian with a full covariance matrix



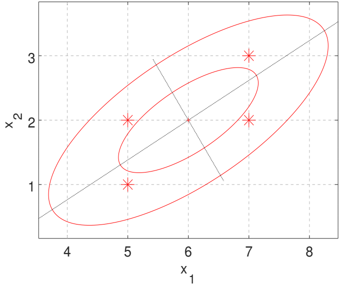


$$\boldsymbol{\mu} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \boldsymbol{\Sigma} = \begin{pmatrix} 1 & -1 \\ -1 & 4 \end{pmatrix} \qquad R = \begin{pmatrix} 1 & -0.5 \\ -0.5 & 1 \end{pmatrix}$$

Example of parameter estimation of a 2D Gaussian

$$\hat{\boldsymbol{\mu}} = \frac{1}{N}\sum_{n=1}^{N} \mathbf{x}_n, \qquad \hat{\Sigma} = \frac{1}{N}\sum_{n=1}^{N} (\mathbf{x}_n - \hat{\boldsymbol{\mu}})(\mathbf{x}_n - \hat{\boldsymbol{\mu}})^T$$

$$\mathbf{x} : \begin{pmatrix} 5 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \end{pmatrix}, \begin{pmatrix} 7 \\ 2 \end{pmatrix}, \begin{pmatrix} 7 \\ 3 \end{pmatrix}$$



$$\boldsymbol{\mu} = \frac{1}{4}\left\{ \begin{bmatrix} 5 \\ 1 \end{bmatrix} + \begin{bmatrix} 5 \\ 2 \end{bmatrix} + \begin{bmatrix} 7 \\ 2 \end{bmatrix} + \begin{bmatrix} 7 \\ 3 \end{bmatrix} \right\} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

$$\mathbf{x}_n - \boldsymbol{\mu} : \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\Sigma = \frac{1}{4}\left\{ \begin{bmatrix} -1 \\ -1 \end{bmatrix} [-1, -1] + \begin{bmatrix} -1 \\ 0 \end{bmatrix} [-1, 0] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} [1, 0] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} [1, 1] \right\} = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Example (*cont.*)

$$\hat{\mu}_i = \frac{1}{N}\sum_{n=1}^{N} x_{ni}, \qquad \hat{\sigma}_{ij} = \frac{1}{N}\sum_{n=1}^{N}(x_{ni} - \hat{\mu}_i)(x_{nj} - \hat{\mu}_j)$$

$$\boldsymbol{x}: \begin{pmatrix}5\\1\end{pmatrix}, \begin{pmatrix}5\\2\end{pmatrix}, \begin{pmatrix}7\\2\end{pmatrix}, \begin{pmatrix}7\\3\end{pmatrix}$$

$$\mu_1 = \tfrac{1}{4}(5+5+7+7) = 6$$

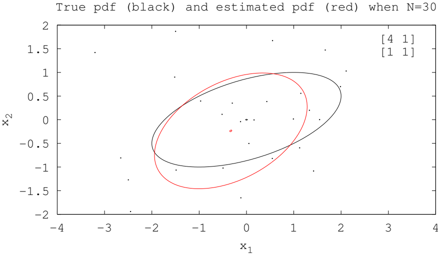
$$\mu_2 = \tfrac{1}{4}(1+2+2+3) = 2$$

$$\boldsymbol{x}-\boldsymbol{\mu}: \begin{pmatrix}-1\\-1\end{pmatrix}, \begin{pmatrix}-1\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}$$

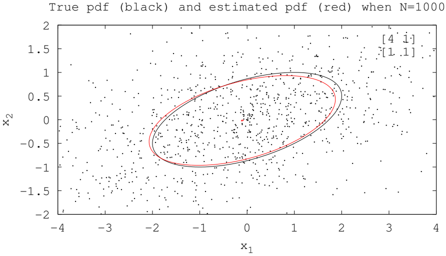
$$\begin{aligned}\boldsymbol{\Sigma}: \quad \sigma_{11} &= \tfrac{1}{4}((-1)^2 + (-1)^2 + 1^2 + 1^2) = 1\\ \sigma_{12} &= \tfrac{1}{4}((-1)\cdot(-1) + (-1)\cdot 0 + 1\cdot 0 + 1\cdot 1) = \tfrac{1}{2}\\ \sigma_{22} &= \tfrac{1}{4}((-1)^2 + 0^2 + 0^2 + 1^2) = \tfrac{1}{2}\end{aligned}$$

Practical issues

Parameter estimation of multivariate Gaussian distribution can be difficult.



$N = 30$



$N = 1000$

Exercise

- Try Q3, Q4, Q5 in Tutorial 3
- Try Q3 in Tutorial 4
- Try Q4 in Tutorial 4, and
  - Find $\boldsymbol{\Sigma}_i^{-1}$ for $i = 1, 2$.
  - Find $|\boldsymbol{\Sigma}_i|$ for $i = 1, 2$.
  - Find the correlation matrix for each class.
  - What the covariance matrix and pdf will be if the naive Bayes assumption is applied?

Exercise *(cont.)*

Additional to Q3 in Tutorial 4:

The sample variance (σ^2_{ML}) is the maximum likelihood estimate for the variance parameter of a one-dimensional Gaussian. Consider the log likelihood of a set of N data points $x_1, \ldots, x_N$ being generated by a Gaussian with the mean μ and variance σ^2.

$$L = \ln p(\{x_1, \ldots, x_N\} \,|\mu, \sigma^2) = -\frac{1}{2}\sum_{n=1}^{N}\left(\frac{(x_n - \mu)^2}{\sigma^2} + \ln\sigma^2 + \ln(2\pi)\right)$$

Assuming that the mean μ is know, show that that maximum likelihood estimate for the variance is indeed the sample variance.

Summary

Gaussians

- Continuous random variable: cumulative distribution function and probability density function
- Univariate Gaussian pdf:

$$p(x \mid \mu, \sigma^2) = N(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$$

- Multivariate Gaussian pdf:

$$p(\mathbf{x} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}) = \frac{1}{(2\pi)^{D/2}|\boldsymbol{\Sigma}|^{1/2}} \exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right)$$

- Estimate parameters (mean and covariance matrix) using maximum likelihood estimation
- Try Lab-6 (next week)