# Inf 2B: Ranking Queries on the WWW

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## Queries

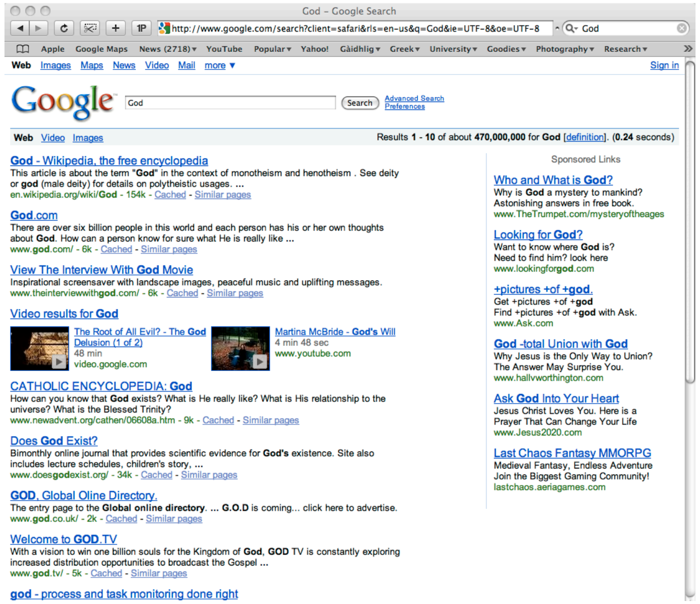
Suppose we have an Inverted Index for a set of webpages.
Disclaimer

- Not *really* the scenario of Lecture 11.
- Indexing for the web is massive-scale: many distributed networks working in parallel.

We search with a term $t$.

Index has many hits for $t$ (say 36,000 for this $t$).
How should we rank them?

## A real search




## Ranking Queries

Inverted Index (probably) stores the frequency of the term $t$ in each document $d$ (e.g., in previous lecture, our index contains $f_{d,t}$ values).

**Idea** Rank answers to queries *in order of frequency of t* in the various webpages.

**Problem** Some great websites will not even contain the term $t$.
For example, there are not many occurrences of the term “University of Edinburgh" on `http://www.ed.ac.uk`

**New Idea** Use structure of web to rank queries.

## Ranking Queries using web structure

Principle:

*Link from one webpage to another confers authority on the target webpage.*

This is the concept behind:

- The Hub-Authority model of Kleinberg.
- PageRank$^{TM}$ ranking system of Google$^{TM}$.
  In early 90s, while PhD students at Stanford, Sergey Brin and Larry Page invented PageRank$^{TM}$ (and founded Google$^{TM}$).

## PageRank$^{TM}$

Webgraph for a particular query:

- vertices $V = [N]$ where $[N] = \{1, 2, \ldots, N\}$ corresponding to pages;
- links are the directed edges of the graph, so $E \subseteq [N] \times [N]$.

Let $G = (V, E)$. Recall:

### Definition

Let $u$ denote some page $u \in [N]$ in the webgraph.

- $In(u)$ is the set of in-edges to $u$. The *in-degree* $in(u)$ is $in(u) = |In(u)|$.
- $Out(u)$ is the set of out-edges from $u$. The *out-degree* $out(u)$ is $out(u) = |Out(u)|$.

## PageRank$^{TM}$

Could use in-degree to measure ranking directly.

But:

- Want pages of high rank to confer *more authority on the pages they link to.*
- *A page with few links should transfer more of its authority to its linked pages than one with many links.*

Assumptions: (for basic PageRank$^{TM}$)

- No “dead-end" pages.
- Every page can hop to every other page via links.
- Aperiodic.

## Principle of PageRank$^{TM}$

Let $R(v)$ denote the *rank of v* for any webpage $v \in [N]$.

For every webpage $u$ in our collection, the following equality should hold:

$$R(u) = \sum_{v \in In(u)} R(v)/out(v)$$

Rank of $u$ is the “total amount of Rank” given from the incoming links to $u$.

## PageRank™ in matrix form

$$(R_1, R_2, \ldots, R_N) \;=\; (R_1, R_2, \ldots, R_N) \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1N} \\ p_{21} & p_{22} & \cdots & p_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ p_{N1} & p_{N2} & \cdots & p_{NN} \end{pmatrix}$$

where

$$p_{uv} \;=\; \begin{cases} 1/out(u), & \text{if } v \in Out(u); \\ 0, & \text{otherwise.} \end{cases}$$

## PageRank™ in matrix form

Shorthand version:

$$R^T = R^T P, \tag{1}$$

where $P = [p_{uv}]_{u,v \in [N]}$ and $R$ is the vector of ranks for $[N]$.
Equivalent to asking for

$$R = P^T R, \tag{2}$$

Looks like condition for $R$ to be an eigenvector of $P^T$ with eigenvalue $\lambda = 1$.

## PageRank™

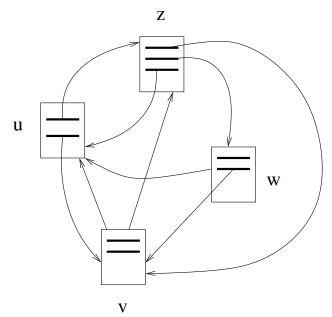
Questions and Answers

- How do we know that 1 is an eigenvalue of the matrix $P^T$?

  Answer: $P^T$ is a stochastic matrix (each column adds to 1), so has eigenvalue 1.
- If 1 *is* an eigenvalue of $P^T$, is it guaranteed to be a *simple* eigenvalue?
  - i.e., any two vectors that satisfy $P^T R = R$ are the same up to a non-zero constant multiple (*linearly dependent*).

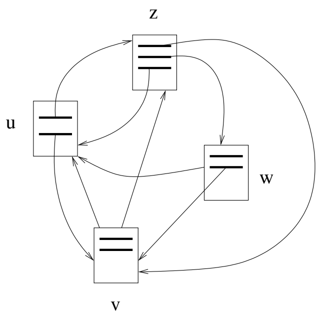
  Answer: Under our assumptions, there is just one linearly independent eigenvector for 1.

## Example



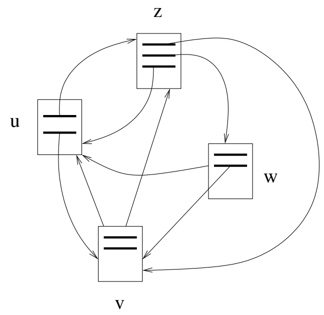

Example webgraph returned by a rare query in ancient times.

## Example




Satisfies all the nice conditions for Basic PageRank™ model (no dead-end pages, can move from any vertex $x$ to any other vertex $y$, aperiodic) .

## Example




$$(R_u, R_v, R_w, R_z) \; = (R_u, R_v, R_w, R_z) \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix}.$$

## Example (continued)

$$(R_u, R_v, R_w, R_z) = (R_u, R_v, R_w, R_z) \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix}.$$

Can “read-off" $R_w = R_z/3$, and propagate this into matrix:

$$(R_u, R_v, R_w, R_z) = (R_u, R_v, R_w, R_z) \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ \frac{1}{3}+\frac{1}{6} & \frac{1}{3}+\frac{1}{6} & \frac{1}{3} & 0 \end{pmatrix}.$$

## Example (continued)

Now remove $R_w$ (keeping $R_w = R_z/3$ to side):

$$(R_u, R_v, R_z) \quad = \quad (R_u, R_v, R_z) \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}.$$

## Example (continued)

$$(R_u, R_v, R_z) \quad = \quad (R_u, R_v, R_z)\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \Longleftrightarrow$$

$$(R_u, R_v - R_z, R_z) \quad = \quad (R_u, R_v, R_z)\begin{pmatrix} 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}.$$

Middle equation reads $R_v - R_z = 1/2(R_z - R_v)$, so $R_v = R_z$.
Final equation says $R_z = 1/2(R_u + R_v)$, so $R_z = R_u$ too.

Solution: $R_u = R_v = R_z$, $R_w = R_z/3$.

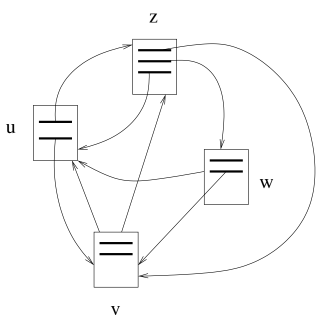
## Alternative (Equivalent) Approach)

Expand vector-matrix product:

$$\begin{aligned} R_u &= \frac{1}{2}R_v + \frac{1}{2}R_w + \frac{1}{3}R_z \\ R_v &= \frac{1}{2}R_u + \frac{1}{2}R_w + \frac{1}{3}R_z \\ R_w &= \frac{1}{3}R_z \\ R_z &= \frac{1}{2}R_u + \frac{1}{2}R_v. \end{aligned}$$

- Subtract the second equation from the first: $R_u - R_v = \frac{1}{2}R_v - \frac{1}{2}R_u$
- It follows that $R_v = R_u$.
- Substituting into the fourth equation: $R_z = R_u$.
- This method is probably preferable for such small examples.

## Example (continued)




Solutions are $R_u = R_v = R_z$, $R_w = R_z/3$, i.e.,

$$(R_u, R_v, R_w, R_z) = c(1, 1, 1/3, 1)$$

where $c$ is a constant.
Not the same as counting in-degree (for this example).

## General PageMark$^{\mathrm{TM}}$ model

- Remove all our assumptions (dead-end pages, connectivity).
- $\lambda$ cannot be assumed to be 1.
- Need to tinker the model. See Lecture Notes.

# Further Reading

Nothing in [GT] or [CLRS].
Papers on the web:

- An Anatomy of a Large-Scale Hypertextual Web Search Engine, by Sergey Brin and Lawrence Page, 1998. Online at: `http://www-db.stanford.edu/ backrub/google.html`
- The PageRank Citation Ranking: Bringing Order to the Web, by Page, Brin, Motwani and Winograd, 1998. Available online from: `http://dbpubs.stanford.edu:8090/pub/1999-66`
- Authoritative Sources in a Hyperlinked Environment, by Jon Kleinberg. Available Online from Jon Kleinberg's webpage: `http://www.cs.cornell.edu/home/kleinber/`