Inf 2B: Graphs, BFS, DFS

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Directed and Undirected Graphs

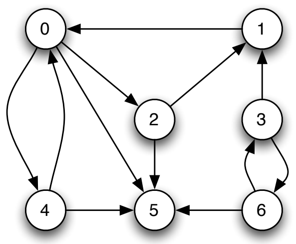
- A *graph* is a mathematical structure consisting of a set of *vertices* and a set of *edges* connecting the vertices.
- Formally: $G = (V, E)$, where V is a set and $E \subseteq V \times V$.
- For edge $e = (u, v)$ we say that e is *directed from u to v*.
- $G = (V, E)$ *undirected* if for all $v, w \in V$:

$$(v, w) \in E \iff (w, v) \in E.$$

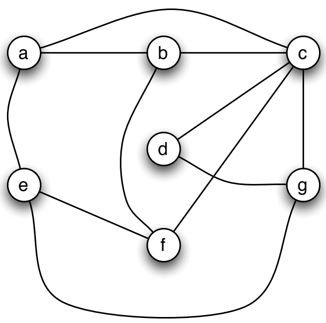
Otherwise *directed*.
Directed $\sim$ *arrows* (one-way)
Undirected $\sim$ *lines* (two-way)
- We assume V is finite, hence E is also finite.

A directed graph

$$\begin{aligned} G &= (V, E), \\ V &= \{0, 1, 2, 3, 4, 5, 6\}, \\ E &= \{(0,2), (0,4), (0,5), (1,0), (2,1), (2,5), \\ &\qquad (3,1), (3,6), (4,0), (4,5), (6,3), (6,5)\}. \end{aligned}$$



An undirected graph



Examples

- *Road Maps.*
  Edges represent streets and vertices represent crossings (junctions).
- *Computer Networks.*
  Vertices represent computers and edges represent network connections (cables) between them.
- *The World Wide Web.*
  Vertices represent webpages, and edges represent hyperlinks.
- . . .

Adjacency matrices

Let $G = (V, E)$ be a graph with n vertices. Vertices of G are numbered $0, \ldots, n-1$.

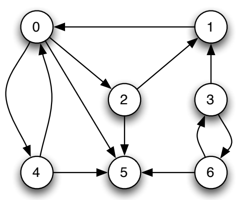
The *adjacency matrix* of G is the $n \times n$ matrix

$$A = (a_{ij})_{0 \le i,j \le n-1}$$

with

$$a_{ij} = \begin{cases} 1, & \text{if there is an edge from vertex } i \text{ to vertex } j; \\ 0, & \text{otherwise.} \end{cases}$$

Adjacency matrix (Example)

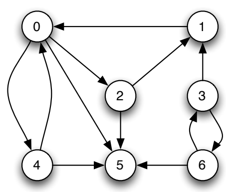


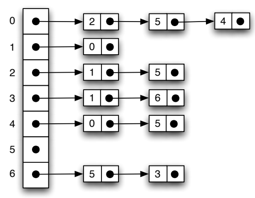
$$\begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

Adjacency lists

Array with one entry for each vertex v, which is a list of all vertices adjacent to v.

Example





Quick Question

Given: graph $G = (V, E)$, with $n = |V|$, $m = |E|$.
For $v \in V$, we write $in(v)$ for in-degree, $out(v)$ for out-degree.

Which data structure has faster (asymptotic) worst-case running-time, for *checking if w is adjacent to v*, for a given pair of vertices?

1. Adjacency list is faster.
2. Adjacency matrix is faster.
3. Both have the same asymptotic worst-case running-time.
4. It depends.

Answer: 2. For an Adjacency Matrix we can check in $\Theta(1)$ time.
An adjacency list structure takes $\Theta(1 + out(v))$ time.

Quick Question

Given: graph $G = (V, E)$, with $n = |V|$, $m = |E|$.
For $v \in V$, we write $in(v)$ for in-degree, $out(v)$ for out-degree.

Which data structure has faster (asymptotic) worst-case running-time, for *visiting all vertices w adjacent to v*, for a given vertex v?

1. Adjacency list is faster.
2. Adjacency matrix is faster.
3. Both have the same asymptotic worst-case running-time.
4. It depends.

Answer: 3. Adjacency matrix requires $\Theta(n)$ time always.
Adjacency list requires $\Theta(1 + out(v))$ time.
In worst-case $out(v) = \Theta(n)$.

Adjacency Matrices vs Adjacency Lists

	adjacency matrix	adjacency list
Space	$\Theta(n^2)$	$\Theta(n + m)$
Time to check if w adjacent to v	$\Theta(1)$	$\Theta(1 + out(v))$
Time to visit all w adjacent to v.	$\Theta(n)$	$\Theta(1 + out(v))$
Time to visit all edges	$\Theta(n^2)$	$\Theta(n + m)$

Sparse and dense graphs

$G = (V, E)$ graph with n vertices and m edges

Observation: $m \leq n^2$

- G *dense* if m close to n^2
- G *sparse* if m much smaller than n^2

Graph traversals

A *traversal* is a strategy for visiting all vertices of a graph while respecting edges.

BFS = breadth-first search

DFS = depth-first search

General strategy:

1. Let v be an arbitrary vertex
2. Visit all vertices reachable from v
3. If there are vertices that have not been visited, let v be such a vertex and go back to (2)

Graph Searching (general Strategy)

Algorithm searchFromVertex(G, v)

```
mark v
put v onto schedule S
while schedule S is not empty do
    remove a vertex v from S
    for all w adjacent to v do
        if w is not marked then
            mark w
            put w onto schedule S
```

Algorithm search(G)

```
ensure that each vertex of G is not marked
initialise schedule S
for all v ∈ V do
    if v is not marked then
        searchFromVertex(G, v)
```


Three colour view of vertices

- Previous algorithm has vertices in one of two states: *unmarked* and *marked*. Progression is

  $$unmarked \longrightarrow marked$$

- Can also think of them as being in one of three states (represented by colours):
  - *White*: not yet seen (not yet investigated).
  - *Grey*: put on schedule (under investigation).
  - *Black*: taken off schedule (completed).

  Progression is

  $$white \longrightarrow grey \longrightarrow black$$

We will use the three colour scheme when studying an algorithm for topological sorting of graphs.

BFS

Visit all vertices reachable from v in the following order:

- v
- all neighbours of v
- all neighbours of neighbours of v that have not been visited yet
- all neighbours of neighbours of neighbours of v that have not been visited yet
- etc.

BFS (using a Queue)

Algorithm bfs(G)

1. Initialise Boolean array *visited*, setting all entries to FALSE.
2. Initialise *Queue* Q
3. **for all** $v \in V$ **do**
4.   **if** $visited[v] = $ FALSE **then**
5.   bfsFromVertex(G, v)

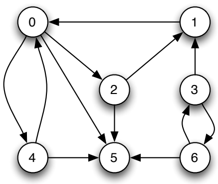
BFS (using a Queue)

Algorithm bfsFromVertex(G, v)

1. $visited[v] = $ TRUE
2. Q.enqueue(v)
3. **while not** Q.isEmpty() **do**
4. $v \leftarrow Q$.dequeue()
5.   **for all** w adjacent to v **do**
6.   **if** $visited[w] = $ FALSE **then**
7. $visited[w] = $ TRUE
8. Q.enqueue(w)

Algorithm bfsFromVertex(G, v)

1. $visited[v] = $ TRUE
2. Q.enqueue(v)
3. **while not** Q.isEmpty() **do**
4. $v \leftarrow Q$.dequeue()
5.   **for all** w adjacent to v **do**
6.   **if** $visited[w] = $ FALSE **then**
7. $visited[w] = $ TRUE
8. Q.enqueue(w)



Quick Question

Given a graph $G = (V, E)$ with $n = |V|$, $m = |E|$, what is the worst-case running time of BFS, in terms of m, n?

1. $\Theta(m + n)$
2. $\Theta(n^2)$
3. $\Theta(mn)$
4. Depends on the number of components.

Answer: 1. To see this need to be careful about bounding running time for the loop at lines 5–8.
Must use the Adjacency List structure.

Answer: 2. if we use adjacency matrix representation.

DFS

Visit all vertices reachable from v in the following order:

- v
- some neighbour w of v that has not been visited yet
- some neighbour x of w that has not been visited yet
- etc., until the current vertex has no neighbour that has not been visited yet
- Backtrack to the first vertex that has a yet unvisited neighbour v'.
- Continue with v', a neighbour, a neighbour of the neighbour, etc., backtrack, etc.

DFS (using a stack)

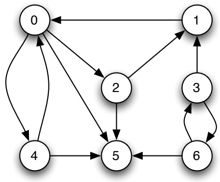
Algorithm dfs(G)

1. Initialise Boolean array *visited*, setting all to FALSE
2. Initialise *Stack* S
3. **for all** $v \in V$ **do**
4.   **if** $visited[v] =$ FALSE **then**
5.   dfsFromVertex(G, v)

DFS (using a stack)

Algorithm dfsFromVertex(G, v)

1. S.push(v)
2. **while not** S.isEmpty() **do**
3. $v \leftarrow S$.pop()
4.   **if** $visited[v] =$ FALSE **then**
5. $visited[v] =$ TRUE
6.   **for all** w adjacent to v **do**
7. S.push(w)



Recursive DFS

Algorithm dfs(G)

1. Initialise Boolean array *visited* by setting all entries to FALSE
2. **for all** $v \in V$ **do**
3.   **if** $visited[v] =$ FALSE **then**
4.   dfsFromVertex(G, v)

Algorithm dfsFromVertex(G, v)

1. $visited[v] \leftarrow$ TRUE
2. **for all** w adjacent to v **do**
3.   **if** $visited[w] =$ FALSE **then**
4.   dfsFromVertex(G, w)

Analysis of DFS

$G = (V, E)$ graph with n vertices and m edges

Without recursive calls:

- dfs(G): time $\Theta(n)$
- dfsFromVertex(G, v): time $\Theta(1 + \text{out-degree}(v))$

Overall time:

$$\begin{aligned}
T(n, m) &= \Theta(n) + \textstyle\sum_{v \in V} \Theta(1 + \text{out-degree}(v)) \\
&= \Theta\Big(n + \textstyle\sum_{v \in V}(1 + \text{out-degree}(v))\Big) \\
&= \Theta\Big(n + n + \textstyle\sum_{v \in V} \text{out-degree}(v)\Big) \\
&= \Theta\Big(n + \textstyle\sum_{v \in V} \text{out-degree}(v)\Big) \\
&= \Theta(n + m)
\end{aligned}$$