Pushdown automata

Informatics 2A: Lecture 10

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Recap of lecture 8

- Context-free languages are defined by context-free grammars.
- A grammar generates strings by applying productions starting from a start symbol.
- This produces a derivation: a sequence of sentential forms ending in a string of terminal symbols.
- Every derivation determines a corresponding syntax tree.
- A grammar is structurally ambiguous if there is a string in its language that can be given two or more syntax trees.

Machines for context-free languages

We've seen that regular languages can be defined by (det. or non-det.) finite automata.

Question: What kinds of machines do context-free languages correspond to?

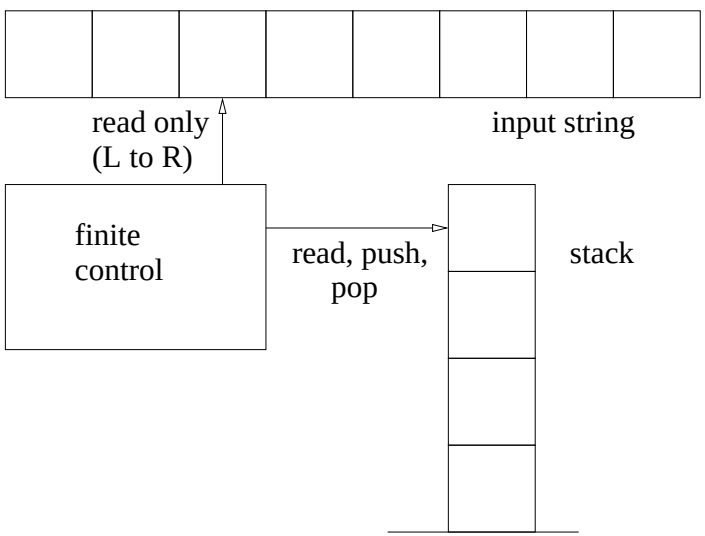
Motivation: Seeing how to recognize sentences of a CF language will be a step towards seeing how to parse such sentences.

Short answer: NFAs equipped with unlimited memory in the form of a stack.

Pushdown automata (PDAs)

As in NFAs, imagine a control unit with finitely many possible states, equipped with a read head that can (only) read the input string from left to right.

Add to this a stack with associated operations of push, pop and read the item on the top of the stack.



Transitions can depend on both the current input symbol and the current stack item.

Transitions can also cause items to be pushed to and/or popped from the stack.

Note: the machine can't read a stack item other than the top one without first popping (and so losing) all items above it.

Example of a PDA

Consider a PDA with a single state q, input alphabet $\Sigma = \{(,)\}$ and stack alphabet $\Gamma = \{(, \bot\}$. Call $\bot$ the initial stack symbol.

Our PDA has four transitions $q \to q$, labelled as follows:

$$^{1}\ (, \bot\ :\ (\bot \qquad ^{2}\ (, (\ :\ ((\qquad ^{3}\ ), (\ :\ \epsilon \qquad ^{4}\ \epsilon, \bot\ :\ \epsilon$$

Meaning. [1] If current read symbol is (and current stack symbol is $\bot$, may pop the $\bot$ and replace it with $(\bot$. Note that we push $\bot$ first, then (, so stack grows to the left.

[2] Similarly with (in place of $\bot$.

[3] If current read symbol is) and current stack symbol is (, may simply pop the (.

[4] If current stack symbol is $\bot$, can just pop it.

Idea: stack keeps track of currently pending ('s. When stack clears, may pop the initial $\bot$ — this ends the computation.

Sample execution

$^1(, \bot : (\bot$ $^2(, (: (($ $^3), (: \epsilon$ $^4\epsilon, \bot : \epsilon$

	Unread input	Stack state
	(()())	$\bot$
$\xrightarrow{1}$	()())	$(\bot$
$\xrightarrow{2}$	)())	$((\bot$
$\xrightarrow{3}$	())	$(\bot$
$\xrightarrow{2}$	))	$((\bot$
$\xrightarrow{3}$	)	$(\bot$
$\xrightarrow{3}$	ϵ	$\bot$
$\xrightarrow{4}$	ϵ	ϵ

Language recognised by example PDA

Our example PDA has a single state q, input alphabet $\Sigma = \{(,)\}$, and stack alphabet $\Gamma = \{(, \perp\}$ with initial stack symbol $\perp$, and transitions $q \to q$,

$$^{1}\ (, \perp\ :\ (\perp \qquad ^{2}\ (, (\ :\ ((\qquad ^{3}\ ), (\ :\ \epsilon \qquad ^{4}\ \epsilon, \perp\ :\ \epsilon$$

This machine can empty its stack at the end of the input string if and only if the input string is a well-matched sequence of brackets.

So the PDA acts as a (non-deterministic) recognizer for the language of well-matched sequences of brackets.

PDAs: formal definition

A (nondeterministic) pushdown automaton (N)PDA M consists of

- a finite set Q of control states
- a finite input alphabet Σ
- a finite stack alphabet Γ including a start symbol $\perp$
- a start state $s \in Q$
- a finite transition relation $\Delta \subseteq (Q \times (\Sigma \cup \{\epsilon\}) \times \Gamma) \times (Q \times \Gamma^*)$

Note that each individual transition has the form $((p, a, s), (q, u))$ where $p, q \in Q$, $a \in \Sigma \cup \{\epsilon\}$, $s \in \Gamma$ and $u \in \Gamma^*$. Such a transition is sometimes drawn as:

$$p \xrightarrow{a,\, s:u} q$$

Accepting by empty stack

A string $x \in \Sigma^*$ is accepted by M if there is some run of M on x, starting at control state s with stack $\perp$, and finishing (at any control state) with empty stack having consumed all of x.

This definition implements acceptance by empty stack.

The language accepted by M is

$$\mathcal{L}(M) = \{x \in \Sigma^* \mid M \text{ accepts } x\}$$

Other acceptance conditions

Other acceptance conditions for PDAs can be found in the literature.

Example variation: Equip M with a set of accepting states, and say a string is accepted if it can trigger a computation ending in an accepting state.

The choice of acceptance condition is not a big deal. For any NPDA of either kind, we can build one of the other kind that accepts the same language. For details see Kozen chapter E.

In Inf2A, we use use empty stack as our default acceptance condition for PDAs.

How powerful are PDAs?

We shall outline the proof of the following theorem:

> *A language L is context-free if and only if there is an NPDA M such that $L = \mathcal{L}(M)$.*

One can also define deterministic pushdown automata (DPDAs) in a reasonable way.

;-(Sadly, there's no analogue of the NFA 'powerset construction' for NPDAs. In fact, there are context-free languages that can't be recognized by any DPDA (example in Lecture 28).

Since DPDAs allow for efficient processing, this prompts us to focus on 'simple' kinds of CFLs that can be recognized by a DPDA. (See next lecture on efficient table-driven parsing.)

From CFGs to NPDAs

Given a CFG with nonterminals N, terminals Σ, productions P and start symbol S.

Build an NPDA with a single state q, input alphabet Σ and stack alphabet $N \cup \Sigma$. Take S as initial stack symbol.

- For each production $X \to \alpha$ in P, include an ϵ-transition

$$q \xrightarrow{\epsilon,\, X:\alpha} q$$

- For each terminal $a \in \Sigma$, include a transition

$$q \xrightarrow{a,\, a:\epsilon} q.$$

Intuition: the stack records a 'guess' at a sentential form for the part of the input still to come.

From CFGs to NPDAs: Example

Recall our grammar for arithmetic expressions:

$$\begin{array}{lcl} \text{Exp} & \to & \text{Var} \mid \text{Num} \mid (\text{Exp}) \\ \text{Exp} & \to & \text{Exp} + \text{Exp} \\ \text{Exp} & \to & \text{Exp} * \text{Exp} \end{array} \qquad \begin{array}{lcl} \text{Var} & \to & x \mid y \mid z \\ \text{Num} & \to & 0 \mid \cdots \mid 9 \end{array}$$

Suppose we turn this into an NPDA as above.
What does an accepting run of this NPDA on the input string (x)*5 look like?

An accepting run of the NPDA

Consider the input string (x)*5.

Our machine can proceed as follows, correctly guessing the rule to apply at each stage.

Transition	Input read	Stack state
	ϵ	Exp
Apply * rule	ϵ	Exp $*$ Exp
Apply () rule	ϵ	(Exp) $*$ Exp
Match (	(	Exp) $*$ Exp
Apply Var rule	(	Var) $*$ Exp
Apply x rule	(	x) $*$ Exp
Match x	(x	) $*$ Exp
Match)	(x)	$*$ Exp
Match *	(x)$*$	Exp
Apply Num rule	(x)$*$	Num
Apply 5 rule	(x)$*$	5
Match 5	(x) $*$ 5	stack empty!

At each stage, combining 'input read' and 'stack state' gives us a sentential form. In effect, the computation traces out a leftmost derivation of t in $\mathcal{G}$. So the computation tells us not just that the string is legal, but how to build a syntax tree for it.

From NPDAs to CFGs: brief sketch

Suppose first M is an NPDA with just one state q.
Can turn M into a CFG: almost the reverse of what we just did.

Use M's stack alphabet Γ as the set of nonterminals of the grammar.

General form for transitions of M is $((q, a, X), (q, \alpha))$, where a and/or α might be ϵ. Turn these into productions $X \to a\alpha$.

Now suppose we have an NPDA M with many states. Can turn it into an equivalent NPDA with just one state, essentially by storing all 'state information' on the stack.

When pushing multiple stack entries, must nondeterministically 'guess' the intermediate states. For details see Kozen chapter 25.

Summary

We've seen that NPDAs exactly 'match' CFGs in terms of their power for defining languages. Indeed, a pushdown store (stack) gives just the computational power needed to deal with nesting.

Accepting computations don't just tell us a string is legal — they 'do parsing' (i.e. tell us how to build a syntax tree).

Problem is that computations here are non-deterministic: must correctly 'guess' which production to apply at each stage. If only things were deterministic, we'd have an efficient parsing algorithm!

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Later, we'll look at 'semi-efficient' parsers that work for *any* CFG.

Reading and prospectus

Relevant reading:

- Core material: Kozen chapters 19, 23, 24.
- Further topics: E, 25, F.

Next time, we look at LL(1) grammars: a class of relatively 'simple' CFGs for which very efficient parsing is possible.