Deterministic finite Automata

Informatics 2A: Lecture 3

Alex Simpson

School of Informatics
University of Edinburgh
als@inf.ed.ac.uk

20th September, 2013

Languages and alphabets

Throughout this course, languages will consist of finite sequences of symbols drawn from some given alphabet.

An alphabet Σ is simply some finite set of *letters* or *symbols* which we treat as 'primitive'. These might be ...

- English letters: $\Sigma = \{a, b, \ldots, z\}$
- Decimal digits: $\Sigma = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
- ASCII characters: $\Sigma = \{0, 1, \ldots, a, b, \ldots, ?, !, \ldots\}$
- Programming language 'tokens': $\Sigma = \{\texttt{if}, \texttt{while}, \texttt{x}, ==, \ldots\}$
- Words in (some fragment of) a natural language.
- 'Primitive' actions performable by a machine or system, e.g. $\Sigma = \{\texttt{insert50p}, \texttt{pressButton1}, \ldots\}$

In toy examples, we'll use simple alphabets like $\{0, 1\}$ or $\{a, b, c\}$.

What is a 'language'?

A language over an alphabet Σ will consist of finite sequences (strings) of elements of Σ. E.g. the following are strings over the alphabet $\Sigma = \{a, b, c\}$:

$$a \qquad b \qquad ab \qquad cab \qquad bacca \qquad cccccccc$$

There's also the empty string , which we usually write as ϵ.

A language over Σ is simply a (finite or infinite) set of strings over Σ. A string s is legal in the language L if and only if $s \in L$.

We write Σ^* for the set of *all* possible strings over Σ. So a language L is simply a subset of Σ^*. ($L \subseteq \Sigma^*$)

(N.B. This is just a technical definition — any *real* language is obviously much more than this!)

Ways to define a language

There are many ways in which we might formally define a language:

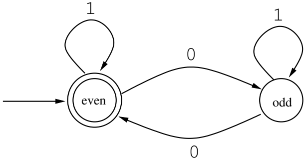
- Direct mathematical definition, e.g.

$$\begin{aligned} L_1 &= \{a, aa, ab, abbc\} \\ L_2 &= \{axb \mid x \in \Sigma^*\} \\ L_3 &= \{a^n b^n \mid n \geq 0\} \end{aligned}$$

- Regular expressions (see Lecture 5).
- Formal grammars (see Lecture 8 onwards).
- Specify some machine for testing whether a string is legal or not.

The more complex the language, the more complex the machine might need to be. As we shall see, each level in the Chomsky hierarchy is correlated with a certain class of machines.

Finite state machines



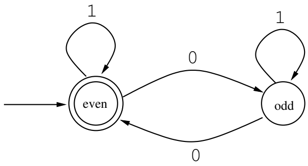
This is an example of a finite state machine over $\Sigma = \{0, 1\}$.

At any moment, the machine is in one of 2 states. For any state, and any symbol in Σ, there's just one 'new state' we can jump to.

The state marked with the in-arrow is picked out as the starting state. So any string in Σ^* gives rise to a sequence of states.

Certain states (with double circles) may be designated as accepting. We call a string 'legal' if it takes us from the start state to some accepting state. In this way, the machine defines a language $L \subseteq \Sigma^*$.

Finite state machines: Clicker question



For the finite state machine shown here, which of the following strings is *not* legal (i.e. not accepted)?

1. ϵ
2. 11
3. 1010
4. 1101

Deterministic finite automata (DFAs)

Formally, a DFA with alphabet Σ consists of:

- A finite set Q of states,
- A transition function $\delta : Q \times \Sigma \to Q$,
- A designated starting state $s \in Q$,
- A set $F \subseteq Q$ of accepting states.

Example:

$$Q = \{\text{even}, \text{odd}\}$$

δ :

	0	1
even	odd	even
odd	even	odd

$$s = \text{even}$$

$$F = \{\text{even}\}$$

The language associated with a DFA

Suppose $M = (Q, \delta, s, F)$ is a DFA with alphabet Σ.
We can define a many-step transition function $\widehat{\delta} : Q \times \Sigma^* \to Q$ by

$$\begin{aligned} \widehat{\delta}(q, \epsilon) &= q \\ \widehat{\delta}(q, xu) &= \delta(\widehat{\delta}(q, x), u) \end{aligned}$$

for $q \in Q$, $x \in \Sigma^*$, $u \in \Sigma$.

Example: $\widehat{\delta}(\mathrm{odd}, 101) = \mathrm{even}$.

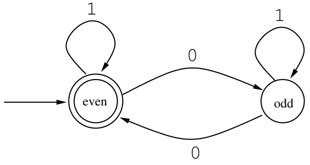
This enables us to define the language accepted by M:

$$\mathcal{L}(M) \;=\; \{x \in \Sigma^* \mid \widehat{\delta}(s, x) \in F\}$$

We call a language $L \subseteq \Sigma^*$ regular if $L = \mathcal{L}(M)$ for *some* DFA M.
Regular languages will occupy us for the next five lectures.

Example

Let M be the DFA shown earlier:



Then $\mathcal{L}(M)$ can be described directly as follows:

$$\mathcal{L}(M) = \{x \in \Sigma^* \mid x \text{ contains an even number of 0's}\}$$

More examples

Which of these three languages do you think are regular?

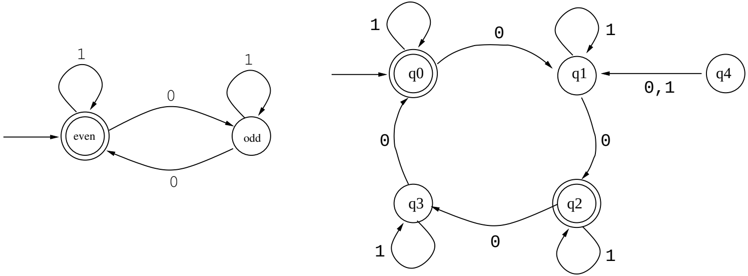
$$
\begin{aligned}
L_1 &= \{a, aa, ab, abbc\} \\
L_2 &= \{axb \mid x \in \Sigma^*\} \\
L_3 &= \{a^n b^n \mid n \geq 0\}
\end{aligned}
$$

If they are, can you design a DFA that shows this?
If not, can you see why not? (We'll revisit this in Lecture 7.)

Aside: DFAs are dead easy to implement and efficient to run. (We don't need much more than a two-dimensional array for the transition function δ.) So it's worth knowing when some task *can* be performed by a DFA.

The Minimization Problem

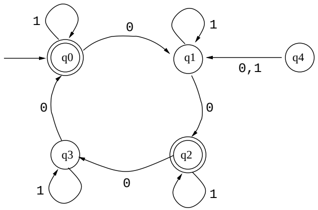
Many different DFAs can give rise to the same language, e.g.:



This raises the question: Is there a 'best' or 'smallest' choice of DFA for a given regular language?

DFA minimization

Suppose $M = (Q, \delta, s, F)$ is the following DFA:



Informally, we can perform the following steps to 'reduce' M:

- Throw away unreachable states (in this case, q4).
- Squish together equivalent states, i.e. states q, q' such that

$$\forall x \in \Sigma^*.\ \widehat{\delta}(q, x) \in F \ \Leftrightarrow\ \widehat{\delta}(q', x) \in F$$

  In this case, q0 and q2 are equivalent, as are q1 and q3.

Let's write Min(M) for the resulting reduced DFA. In this case, Min(M) is essentially the two-state machine on the previous slide.

Properties of minimization

The minimization operation on DFAs has the following pleasing properties:

- $\mathcal{L}(\mathrm{Min}(M)) = \mathcal{L}(M)$.
- $\mathrm{Min}(M) \cong \mathrm{Min}(M')$ if and only if $\mathcal{L}(M) = \mathcal{L}(M')$.
- (Consequence.) $\mathrm{Min}(\mathrm{Min}(M)) \cong \mathrm{Min}(M)$.
- $\mathrm{Min}(M)$ is the smallest DFA (in terms of number of states) with the same language as M.

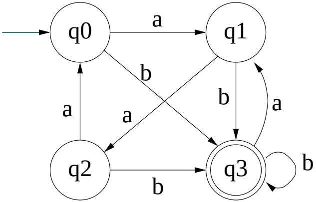
Here '$\cong$' means the two DFAs are isomorphic: that is, the same apart from a possible renaming of states.

So up to isomorphism, minimization gives a 'best', or standard, choice of a DFA for a given regular language.

For a formal treatment of minimization, see Kozen chapters 13–16.

Clicker question

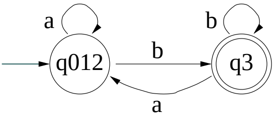
Consider the following DFA over $\{a, b\}$.



How many states does the minimized DFA have? 1, 2, 3 or 4?

Solution

The minimized DFA has just 2 states! q0, q1 and q2 can all be identified, but clearly q3 must be kept distinct from these.



Notice that the corresponding language consists of *all strings ending with b*.

Minimization in practice

Let's look again at our definition of equivalence of states:

$$q \approx q' \quad \text{iff} \quad \forall x \in \Sigma^*.\ \widehat{\delta}(q,x) \in F \ \Leftrightarrow\ \widehat{\delta}(q',x) \in F$$

This is fine as an abstract mathematical definition of equivalence, but it doesn't seem to give us a way to compute which states are equivalent: we'd have to 'check' infinitely many strings $x \in \Sigma^*$.

Fortunately, there's an actual algorithm for DFA minimization that works in reasonable time.

This is useful in practice: we can specify our DFA in the most convenient way without worrying about its size, then minimize to a more 'compact' DFA to be implemented e.g. in hardware.

An algorithm for minimization

First eliminate any unreachable states (easy).

Then create a table of all possible pairs of states (p, q), initially unmarked. (E.g. a two-dimensional array of booleans, initially set to false.) We mark pairs (p, q) as and when we discover that p and q *cannot* be equivalent.

1. Start by marking all pairs (p, q) where $p \in F$ and $q \notin F$, or vice versa.
2. Look for unmarked pairs (p, q) such that for some $u \in \Sigma$, the pair $(\delta(p, u), \delta(q, u))$ is marked. Then mark (p, q).
3. Repeat step 2 until no such unmarked pairs remain.

If (p, q) is still unmarked, can collapse p, q to a single state.

Illustration of minimization algorithm

Consider the following DFA over $\{a, b\}$.

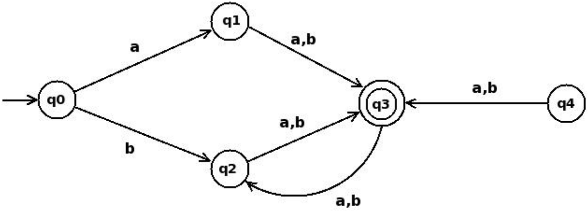


Illustration of minimization algorithm

After eliminating unreachable states:

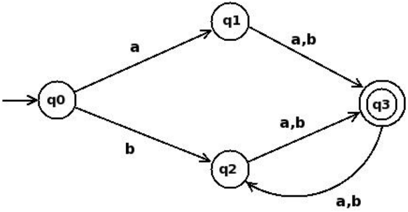
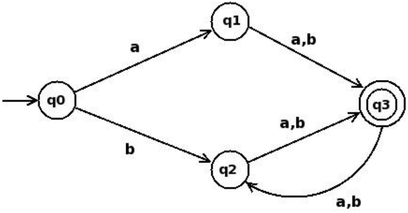


Illustration of minimization algorithm

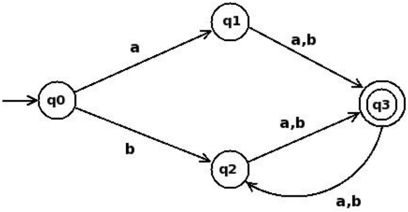
We mark states to be kept distinct using a half matrix:



q0				
q1	.			
q2	.	.		
q3	.	.	.	
	q0	q1	q2	q3

Illustration of minimization algorithm

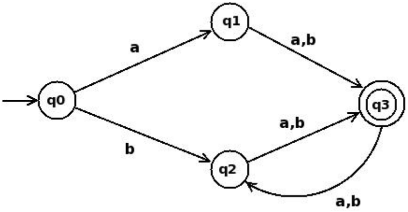
First mark accepting/non-accepting pairs:



	q0	q1	q2	q3
q0				
q1	·			
q2	·	·		
q3	✓	✓	✓	

Illustration of minimization algorithm

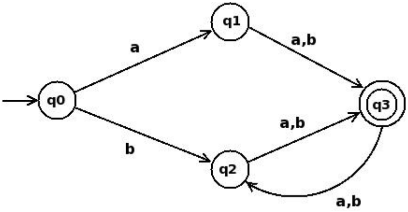
(q0,q1) is unmarked, qo $\xrightarrow{a}$ q1, q1 $\xrightarrow{a}$ q3, and (q1,q3) is marked.



q0				
q1	·			
q2	·	·		
q3	✓	✓	✓	
	q0	q1	q2	q3

Illustration of minimization algorithm

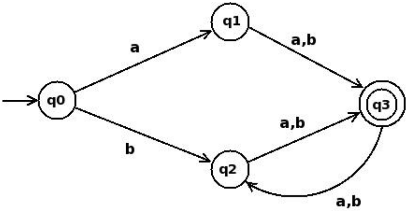
So mark (q0,q1).



	q0	q1	q2	q3
q0				
q1	✓			
q2	·	·		
q3	✓	✓	✓	

Illustration of minimization algorithm

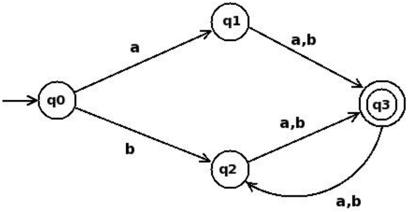
(q0,q2) is unmarked, qo $\xrightarrow{a}$ q1, q2 $\xrightarrow{a}$ q3, and (q1,q3) is marked.



	q0	q1	q2	q3
q0				
q1	✓			
q2	·	·		
q3	✓	✓	✓	

Illustration of minimization algorithm

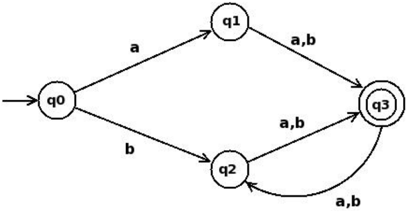
So mark (q0,q2).



q0				
q1	✓			
q2	✓	.		
q3	✓	✓	✓	
	q0	q1	q2	q3

Illustration of minimization algorithm

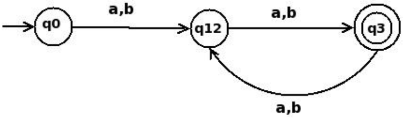
The only remaining unmarked pair (q1,q2) stays unmarked.



q0				
q1	✓			
q2	✓	·		
q3	✓	✓	✓	
	q0	q1	q2	q3

Illustration of minimization algorithm

So obtain mimized DFA by collapsing q1, q2 to a single state.



Reading

Relevant reading:

- DFAs and regular languages: Kozen chapter 3; J & M section 2.2.
- Minimization: Kozen chapters 13–16.

Next time: Non-deterministic finite automata (NFAs), and relationship with DFAs. (See rest of Kozen chapters 5 and 6.)