UNIVERSITY OF EDINBURGH

COLLEGE OF SCIENCE AND ENGINEERING

SCHOOL OF INFORMATICS

INFORMATICS 1 — COMPUTATION & LOGIC

Tuesday 1st April 2014

00:00 to 00:00

INSTRUCTIONS TO CANDIDATES

1. **Note that ALL QUESTIONS ARE COMPULSORY.**
2. **DIFFERENT QUESTIONS MAY HAVE DIFFERENT NUMBERS OF TOTAL MARKS. Take note of this in allocating time to questions.**
3. **CALCULATORS MAY NOT BE USED IN THIS EXAMINATION.**

THIS EXAMINATION WILL BE MARKED ANONYMOUSLY

1. This question concerns the 64 possible truth valuations of six propositional letters, $ABCDEF$. For each of the following expressions say how many of the 64 valuations satisfy the expression:

   Use the space provided for any rough working, and to briefly explain your reasoning.

   (a) $(A?B : C)$

   Answer: 32

   Reason:

A	B	C	$A?B : C$
$\top$	$\top$	$\top$	$\top$
$\top$	$\top$	$\bot$	$\top$
$\top$	$\bot$	$\top$	$\bot$
$\top$	$\bot$	$\bot$	$\bot$
$\bot$	$\top$	$\top$	$\top$
$\bot$	$\top$	$\bot$	$\bot$
$\bot$	$\bot$	$\top$	$\top$
$\bot$	$\bot$	$\bot$	$\bot$

   Four valuations of A, B, C are satisfying, D, E, F has eight, $4 \times 8 = 32$

   [*3 marks*]

   (b) $(A \to B) \to C$

   Answer: 40

   Reason:

A	B	C	$A \to B$	$(A \to B) \to C$
$\top$	$\top$	$\top$	$\top$	$\top$
$\top$	$\top$	$\bot$	$\top$	$\bot$
$\top$	$\bot$	$\top$	$\bot$	$\top$
$\top$	$\bot$	$\bot$	$\bot$	$\top$
$\bot$	$\top$	$\top$	$\top$	$\top$
$\bot$	$\top$	$\bot$	$\top$	$\bot$
$\bot$	$\bot$	$\top$	$\top$	$\top$
$\bot$	$\bot$	$\bot$	$\top$	$\bot$

   Five valuations of A, B, C are satisfying $5 \times 8 = 40$

   [*3 marks*]

(c) $A \to (B \to C) \wedge (D \vee E \vee F)$

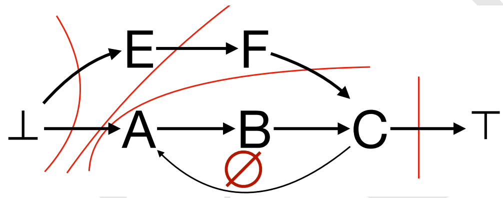
Answer: 53

Reason:
This should be read as $A \to ((B \to C) \wedge (D \vee E \vee F))$
This is true if A is false or $(B \to C) \wedge (D \vee E \vee F)$ is true.
There are $3 \times 7 = 21$ valuations of B, C, D, E, F that make the latter true.
We can add these with A true, to the $2^5 = 32$ valuations of A, B, C, D, E, F the make A false, to give the answer, 53

[*3 marks*]

(d) $(A \to B) \wedge (B \to C) \wedge (C \to A) \wedge (E \to F) \wedge (F \to C)$

Answer: 8

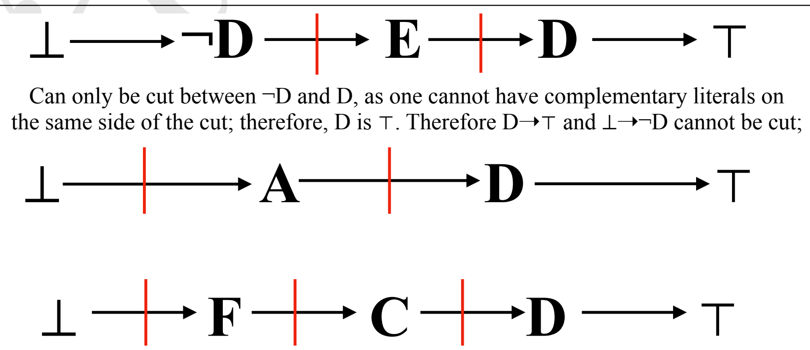


Reason:
4 satisfying valuations of A, B, C, E, F; 2 of D

[*3 marks*]

(e) $(A \to D) \wedge (C \to D) \wedge (\neg D \to E) \wedge (E \to D) \wedge (F \to C)$

Answer: 24



Reason:
A, C, D, E, F has $2 \times 2 \times 3 = 12$, B has 2.

[*3 marks*]

2. For each of the following entailments complete two Karnaugh maps, one to represent the assumption and one the conclusion, by **marking the valuations that make the expression false**.

   Place a mark in one of the check boxes provided, to indicate whether the entaiment is valid. Give a reason for your answer in the box provided.

   Use your Karnaugh maps to give a simple CNF for each assumption.

   (a) $\neg(\neg A \wedge \neg C) \wedge (B \to \neg C) \models \neg(B \to A)$ Valid ☐ Invalid ☑ [*1 mark*]

assumption		CD 00	01	11	10
AB	00	•	•		
	01	•	•	•	•
	11			•	•
	10				

conclusion		CD 00	01	11	10
AB	00	•	•	•	•
	01				
	11	•	•	•	•
	10	•	•	•	•

[*4 marks*]

Reason: Eight valuations that are not excluded by the premises are excluded by the conclusion. It is therefore possible for the assumption to be true while the conclusion is false; for example, any valuation in which A and $\neg B$ is a counterexample.

assumption CNF: $(A \vee C) \wedge (\neg B \vee \neg C)$

[*5 marks*]

   (b) $(A \oplus B) \to (C \to D) \models (A \to C) \to (A \to (\neg B \to D))$ Valid ☑ Invalid ☐ [*1 mark*]

assumption		CD 00	01	11	10
AB	00				
	01				•
	11				
	10				•

conclusion		CD 00	01	11	10
AB	00				
	01				
	11				
	10				•

[*4 marks*]

Reason: The one state excluded by the conclusion is also excluded by the assumption. Therefore, there exists no valuation that makes the assumption true but the conclusion false

assumption CNF: $(A \vee \neg B \vee \neg C \vee D) \wedge (\neg A \vee B \vee \neg C \vee D)$

[*5 marks*]

3. (a) Convert each of the following expressions to CNF

• $R \to (P \wedge Q)$	$(\neg R \vee P) \wedge (\neg R \vee Q)$	[*2 marks*]
• $(S \oplus P \ ?\ T\ :\ R)$	$(\neg S \vee P \vee T) \wedge (\neg P \vee S \vee T) \wedge (S \vee P \vee R) \wedge (\neg S \vee \neg P \vee R)$ see addendum for working...	[*2 marks*]
• $(T \vee P \ ?\ Q : R)$	$(\neg T \vee Q) \wedge (\neg P \vee Q) \wedge (T \vee P \vee R)$	[*2 marks*]
• $\neg(A \leftrightarrow B) \to C$	$(\neg A \vee B \vee C) \wedge (A \vee \neg B \vee C)$	[*2 marks*]

(b) Use resolution to determine whether the entailment
$(B \to A) \to (A \to C) \ \vdash \ B \to C$
is valid, and produce a counterexample if it is not. [*4 marks*]

$\{\{\neg A, B, C\}, \{\neg A, C\}, \{B\}, \{\neg C\}\}$

	A	B	C	
$^C\cancel{\{\neg A, B, C\}}$	-	-		$\{\neg A\}$
$^C\cancel{\{\neg A, C\}}$				$\{\neg A, B\}$
$\{B\}$				
$^C\cancel{\{\neg C\}}$				

Answer invalid

Counterexample? $\neg A, B, \neg C$ [*2 marks*]

(c) Use resolution to determine whether
$(P \vee Q) \to (R \vee S),\ Q \to \neg R \ \vdash \ Q \to S$
is valid and produce a counter-example if it is not. [*4 marks*]

$\{\{\neg P, R, S\}, \{\neg Q, R, S\}, \{\neg Q, \neg R\}, \{Q\}, \{\neg S\}\}$

	P	Q	R	S	
$^R\cancel{\{\neg P, R, S\}}$	-	$^R\cancel{\{R, S\}}$	$^\dagger\{\neg P, S\}$	$\{\}$	
$^Q\cancel{\{\neg Q, R, S\}}$		$^R\cancel{\{\neg R\}}$	$^S\cancel{\{S\}}$		
$^Q\cancel{\{\neg Q, \neg R\}}$					
$^Q\cancel{\{Q\}}$					
$^S\cancel{\{\neg S\}}$					

Answer valid

Counterexample? - [*2 marks*]

† NOTE: Once we derive the empty clause we stop: no need to resolve with $\{\neg P, S\}$.

Gentzen Rules

Question 4 refers to these rules.

$$\frac{}{\Gamma, A \vdash \Delta, A}\ (I)$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ (\wedge L) \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ (\vee R)$$

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\ (\vee L) \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\ (\wedge R)$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta}\ (\to L) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}\ (\to R)$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\ (\neg L) \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ (\neg R)$$

A and B are propositional expressions, Γ, Δ are sets of expressions, and Γ, A refers to $\Gamma \cup \{A\}$.

4. Use the Gentzen rules, provided on the previous page, to attempt to prove the following entailment, your **goal**:

$$(P \to Q) \to R,\ S \vee P \ \vdash\ \neg R \to (Q \to S) \qquad \text{(goal)}$$

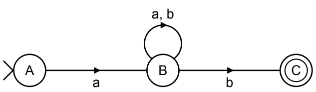
(a) Which of the rules have a conclusion matching this goal?
For each such rule complete a line in the table below showing the name of the rule and the bindings for Γ, Δ, A, B [*10 marks*]

Rule	Γ	Δ	A	B
$\to L$	$S \vee P$	$\neg R \to (Q \to S)$	$P \to Q$	R
$\vee L$	$(P \to Q) \to R$	$\neg R \to (Q \to S)$	S	P
$\to R$	$(P \to Q) \to R,\ S \vee P$	$\emptyset$	$\neg R$	$Q \to S$

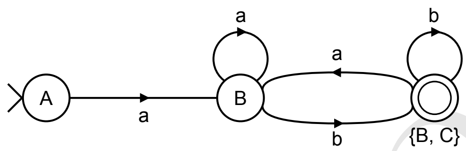
(b) Use the Gentzen rules to construct a formal proof with the goal as conclusion.
Label each step in your proof with the name of the rule being applied. [*10 marks*]

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(I)}
\UnaryInfC{$(P \to Q) \to R, S, Q \ \vdash R, S$}
\AxiomC{}
\RightLabel{(I)}
\UnaryInfC{$P, Q \ \vdash Q, R, S$}
\RightLabel{$(\to R)$}
\UnaryInfC{$P, Q \ \vdash P \to Q, R, S$}
\AxiomC{}
\RightLabel{(I)}
\UnaryInfC{$R, P, Q \ \vdash R, S$}
\RightLabel{$(\to L)$}
\BinaryInfC{$(P \to Q) \to R, P, Q \ \vdash R, S$}
\RightLabel{$(\vee L)$}
\BinaryInfC{$(P \to Q) \to R, S \vee P, Q \ \vdash R, S$}
\RightLabel{$(\to R)$}
\UnaryInfC{$(P \to Q) \to R, S \vee P \ \vdash R, Q \to S$}
\RightLabel{$(\neg L)$}
\UnaryInfC{$(P \to Q) \to R, S \vee P, \neg R \ \vdash Q \to S$}
\RightLabel{$(\to R)$}
\UnaryInfC{$(P \to Q) \to R,\ S \vee P \ \vdash\ \neg R \to (Q \to S)$}
\end{prooftree}
$$

5. Give a regular expression (`re`) for the language accepted by each FSM
Mark the check boxes to show the strings it accepts, and whether it, together with any implicit black hole state, is deterministic.
Draw an equivalent DFA if it is not.

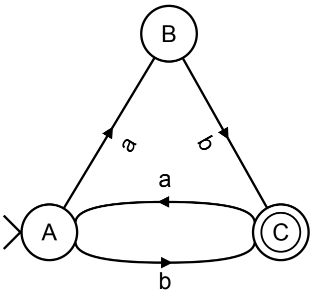
(a) 

abba ☐
abab ☒
abbb ☒
abba ☐
DFA ☐



re: aa*b(aa*b|b)* or a(a|b)b

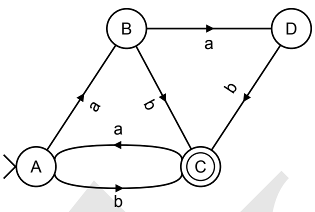
[*5 marks*]

(b) 

abba ☐
abab ☒
abbb ☐
baab ☒
DFA ☒

re: (b|ab)(ab|aab)*

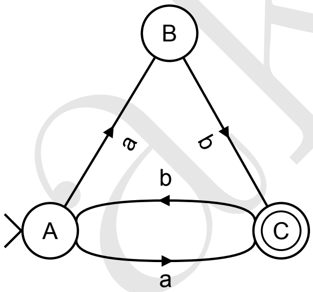
[*5 marks*]

(c) 

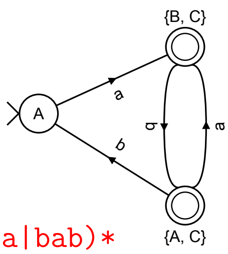
abab ☒
abba ☐
babb ☐
baba ☐
DFA ☒

re: (b|ab|aab)(ab|aab|aaab)*

[*5 marks*]

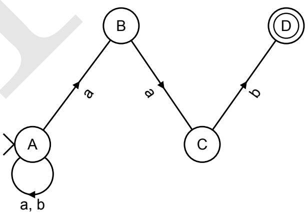
(d) 

aaa ☐
baab ☐
bbab ☐
bbbb ☐
DFA ☐

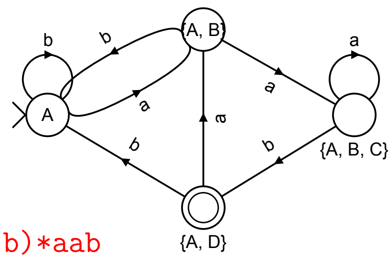


re: (a|ab)(ba|bab)*

[*5 marks*]

(e) 

aaab ☒
aab ☒
baab ☒
bbbb ☐
DFA ☐



re: (a|b)*aab

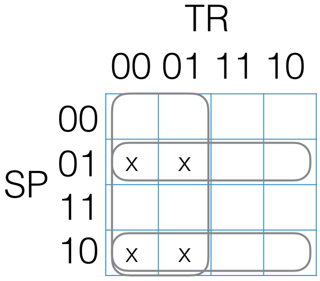
[*5 marks*]

Addendum

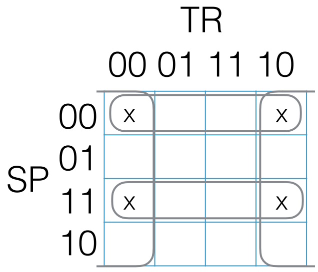
Convert the following expression to CNF: $(S \oplus P \ ?\ T \ :\ R)$. For examples like this one from 3(a), it may be helpful to use a Karnaugh map.

The first step is to mark all the states in which the expression is false. This expression is is false if either $(S \oplus P) \wedge \neg T$ or $\neg(S \oplus P) \wedge \neg R$.

It is easy to mark the states corresponding to each of these. For example, the $(S \oplus P) \wedge \neg T$ is true in the SP row labeled 01 and the row labelled 10, and $\neg T$ is true in the coulmns lablled with 00 or 01.

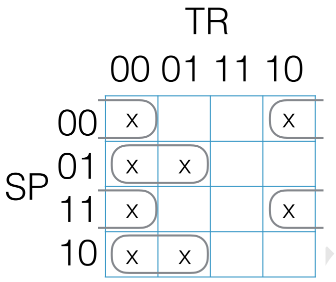


$(S \oplus P) \wedge \neg T$



$\neg(S \oplus P) \wedge \neg R$

Combining these gives us the states in which $\neg(S \oplus P \ ?\ T \ :\ R)$.



$\neg(S \oplus P \ ?\ T : R)$

On this diagram we have marked four blocks, each of which corresponds to a clause in the CNF – we just need enough blocks of excluded states to cover all the excluded states.

For example, the block including states 0000 and 0010 (the two top corners of the Karnaugh map. is excluded by the clause $S \vee P \vee R$.

The remaining blocks, working down the diagram, are excluded by $S \vee \neg P \vee T$, $\neg S \vee \neg P \vee R$, $\neg S \vee P \vee T$.

These four clauses give our clausal form. We could also add three other blocks included in the excluded states, but these are redundant, since they follow from the four we have – they can be derived from the Karnaugh map or derived from the four we have using resolution.