

Original expression:

$$P \rightarrow (R \rightarrow S), Q \rightarrow \neg(R \wedge S) \rightarrow (P \vee \neg B) \vdash (R \vee S) \rightarrow \neg(P \vee \neg Q)$$

To see if the expression is valid we check if we can satisfy an expression where all premises are true and the conclusion is false (which is equal to expression that is a conjunction of premises and negation of conclusion, where we aim to make all of them true):

$$P \rightarrow (R \rightarrow S), Q \rightarrow \neg(R \wedge S) \rightarrow (P \vee \neg B), \neg((R \vee S) \rightarrow \neg(P \vee \neg Q))$$

Conversion to CNF: here I convert each part of the conjunction separately to avoid rewriting:

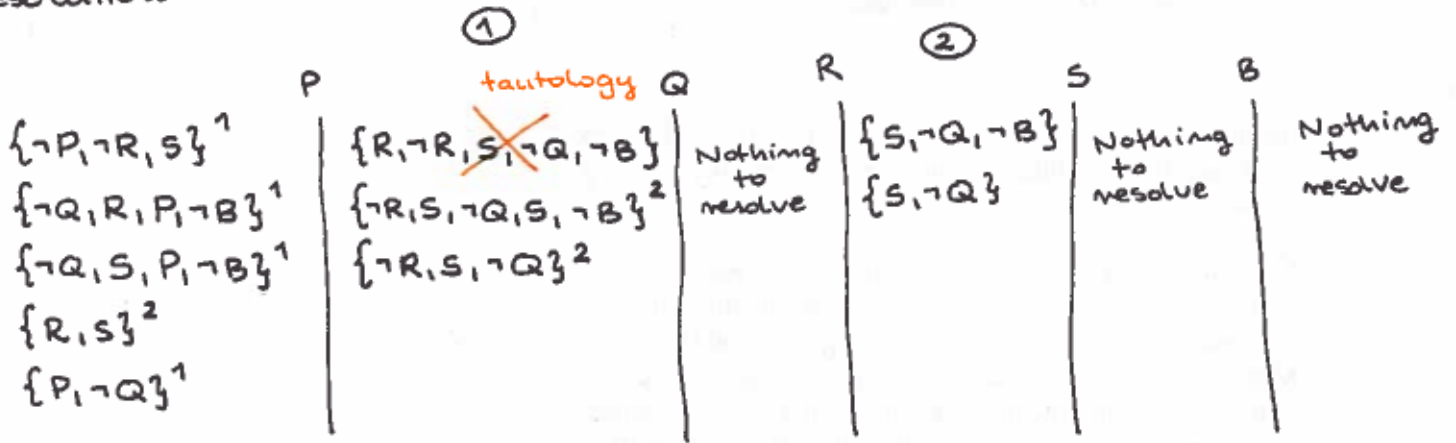
$$\begin{aligned} P \rightarrow (R \rightarrow S) &\equiv \neg P \vee (R \rightarrow S) \equiv \neg P \vee (\neg R \vee S) \equiv \neg P \vee \neg R \vee S \\ &\equiv \{\neg P, \neg R, S\} \\ Q \rightarrow (\neg(R \wedge S) \rightarrow (P \vee \neg B)) &\equiv \neg Q \vee (\neg(R \wedge S) \rightarrow (P \vee \neg B)) \equiv \leftarrow \boxed{\rightarrow \text{ is right-associative}} \\ &\equiv \neg Q \vee ((R \wedge S) \vee (P \vee \neg B)) \equiv \leftarrow \boxed{((A \wedge B) \vee C) \equiv ((A \vee C) \wedge (B \vee C))} \\ &\equiv \neg Q \vee ((R \vee P \vee \neg B) \wedge (S \vee P \vee \neg B)) \equiv \leftarrow \boxed{\vee \text{ is distributive}} \\ &\equiv (\neg Q \vee R \vee P \vee \neg B) \wedge (\neg Q \vee S \vee P \vee \neg B) \\ &\equiv \{\neg Q, R, P, \neg B\}, \{\neg Q, S, P, \neg B\} \end{aligned}$$

$$\begin{aligned} \neg((R \vee S) \rightarrow \neg(P \vee \neg Q)) &\equiv \neg(\neg(R \vee S) \vee \neg(P \vee \neg Q)) \equiv \\ &\equiv (R \vee S) \wedge (P \vee \neg Q) \\ &\equiv \{R, S\}, \{P, \neg Q\} \end{aligned}$$

Full CNF:

$$\{\{\neg P, \neg R, S\}, \{\neg Q, R, P, \neg B\}, \{\neg Q, S, P, \neg B\}, \{R, S\}, \{P, \neg Q\}\}$$

Resolution :



(Numbers next to clauses refer to the step in which the clause was resolved)

Counterexamples:

- 1) $S, \neg Q, R, P, B$
- 2) $S, \neg Q, R, P, \neg B$
- 3) $S, \neg Q, R, \neg P, B$
- 4) $S, \neg Q, R, \neg P, \neg B$
- 5) $S, \neg Q, \neg R, P, B$
- 6) $S, \neg Q, \neg R, P, \neg B$
- 7) $S, \neg Q, \neg R, \neg P, B$
- 8) $S, \neg Q, \neg R, \neg P, \neg B$

- 9) S, Q, P, R, B
- 10) $S, Q, P, R, \neg B$
- 11) $S, Q, P, \neg R, B$
- 12) $S, Q, P, \neg R, \neg B$

- 13) $\neg S, \neg Q, R, \neg P, B$
- 14) $\neg S, \neg Q, R, \neg P, \neg B$

Combination of S and $\neg Q$ makes all the clauses in the resolution true, so for remaining variables, all truth combinations are allowed

Here S, Q, P makes all clauses true

$\neg S, \neg Q, R, \neg P$ makes all clauses true so we are free to pick any value for B

Since we resolved all clauses we could and did not derive the empty clause, our expression is satisfiable.

Counterexamples are valuations of the expression that make it true. Each counterexample makes the premises true and the negation of the conclusion true, which is equal to making premises true and conclusion false, proving that the entailment is invalid.

Note that only one counterexample is necessary, here all were shown for reference.